ELEMENTS OF MECHANICS.

THE

ELEMENTS OF MECHANICS:

BY

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AND CHEMISTRY

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TO

CLEMENT C. MOORE, LL.D.,

SENIOR TRUSTEE OF COLUMBIA COLLEGE,

THIS WORK IS RESPECTFULLY INSCRIBED,

AS A TOKEN OF GRATITUDE FOR MANY

AND SIGNAL FAVOURS,

AND

OF HIGH RESPECT

FOR HIS WORTH AND VIRTUES.

PREFACE.

The work which is now submitted to the public, comprises a portion of the materials collected for the courses, it is my duty annually to deliver in Columbia College. It was originally intended that the subject of Practical Mechanics, should have accompanied the Elements; thus forming a full treatise on the theoretic and practical parts of that useful and interesting branch of Science. It was however found that in this way the work would have assumed too bulky a form. The applications of the elementary principles of the present work to the construction of Machines, have therefore been withheld, until the sense of the public be declared as to its merits. Should the verdict be favourable, the author may be encouraged to proceed with the publication, not only of the sequel to these elements, but with that of some of the other subjects to which a labour of twelve years have been devoted, namely: Pure Physics, Chemistry applied to the Arts, and Practical Astronomy.

In the discussion of subjects so extensive and various, the author is aware, that he has been denied the advantages that are to be derived from the division of labour, and has been unable to devote to any one object, that steady attention, that can alone ensure entire success. In spite of these disadvantages, he ventures to submit the present work to the public, in the belief that it cannot fail to be useful to the student of Mechanical Science. To those who have already made progress, this volume may present little novelty; all that it contains of most value will be readily traced to obvious, if not familiar sources. But to the learner, he cannot but hope that it will offer in a condensed, and generally in a simple and almost popular form, facts, principles, and methods of investigation, that he will find in no single work in any language, and which must be sought for in various treatises, most of them inaccessible to those who read no other language but the English.

The work presents the mixture of strict mechanical principles, with the physical inferences from experiment and observation, that is demanded by the plan of teaching Natural Philosophy, which is generally adopted in this country, and which is habitual in most of the English Treatises on that general subject. It is therefore a combination of the subject, styled by the French *Mecanique*, with so much of the department called by them *Physique*, as is necessary for its illustration, and for preparing the way for practical applications. Nor have the latter been wholly omitted. They have, in the scope of the treatise, frequently come into view, and have in all such cases been treated of in a concise manner.

In the use of the term "Mechanics", it has been employed as including the whole science of Equilibrium and Motion, and therefore as comprising the departments of Hydrostaticks and Hydrodynamicks.

One object has been kept steadily in view, namely, that the student shall not be compelled, after having mastered this treatise, to renew his elementary studies, in case he should wish to rise to the higher applications of mechanical science. Should the author not have failed in this, the present work, however humble, in the limitation of its direct applications to the mere works of human art, may serve as an introduction to the science that investigates the mechanism of the universe.

In order to obviate the necessity of continually quoting the authorities for the various facts and investigations employed in the work, by which the text would have encumbered, or the margin loaded with notes, a list of the works that have been most frequently made use of is subjoined. In some few cases, where the investigations are copied without alteration, or where they are not complete, the passage of the author quoted, is expressly referred to in a note. This list of authorities is far from complete; to have made it so would have appeared rather as a parade of research, than as a security from the charge of quoting without proper acknowledgment; it has therefore been principally confined to those writers whose labours have not become in some measure the common property of all who follow them in this department of science.

COLUMBIA COLLEGE, New-York, *February 1st*, 1832.

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CORRIGENDA.

On Page 3 line 22, *for* "$\cos.^2 a+\cos.^2 b+\cos.^2 c=0$," *read* "$\cos.^2 a+\cos.^2 b+\cos.^2 c=1$."

14 line 12, *for* "$\cos.^2 a+\cos.^2 b+\cos.^2 c=0$," *read* "$\cos.^2 a+\cos.^2 b+\cos.^2 c=1$."

42 line 24, *for* "$t=\frac{\pi}{v-v'}+\frac{\pi}{v'-v''}$," *read* "$t=\frac{\pi}{v-v'}\times\frac{\pi}{v'-v''}$."

54 line 31, *for* "g" *read* "y."

112 line 1, *for* "CHAP. V" *read* "CHAP. IV."

160 line 7, *for* "$P=\frac{W'd}{c}$," *read* "$P=\frac{W'd}{C}$."

line 12, *for* "$P=\frac{Wdb}{ca}$," *read* "$P=\frac{Wdb}{Ca}$."

270 (291) *for* "$1 : 1+\frac{1}{2}h$," *read* "$1+\frac{1}{2}h : 1$."

320 (355) *for* "$h=\frac{a}{D^2}$," *read* "$h=\frac{a^2}{D}$."

327 line 10, *for* "substances," *read* "surfaces."

333 line 13, *fill in the blank*, "(   )," *with* "(299)."

422 line 31, *for* "wv," *read* "αv."

449 line 4, *for* "friction," *read* "function."

437 line 53, *for* "level," *read* "lever."

BOOK I.

OF EQUILIBRIUM.

CHAPTER I.

General Principles.

1. Mechanics is the department of Physical Science which treats of Motion, and of the construction of Machines.

2. Bodies appear to us to be in motion, when they change their position in respect to other bodies that we conceive to be at rest; but even bodies that appear to our earlier investigation to be absolutely at rest, may be in a state of rapid motion, as, for instance, the body of the earth itself; which, to the uneducated and ignorant, appears solid and immovable, although it can be shown by scientific proofs to be a state of rapid motion, both of rotation and translation. Hence we refer motion, in the abstract, to unbounded and immovable space. As space is extended in three dimensions, a body may move in the direction of any one of them, or may have a motion intermediate between two or more of them: it may rise or fall, approach or recede, pass to the right or to the left, or its motion may be combined of two, or all three of those varieties.

In order to render these circumstances of motion definite, we refer the position of a point to three planes, supposed to be immovable in absolute space, and which cut each other at right angles.

3. The perpendicular distances from the point to these three planes, are called its Co-ordinates; the mutual intersection of any two of the planes is called an Axis, and the common intersection of three planes is called the Origin of the co-ordinates. Each of the axes is parallel to one of the co-ordinates, for the common intersection of two of the planes is perpendicular to the third,

and the co-ordinates are by definition each drawn perpendicular to one of the planes.

4. If the point be in motion, it will be shown by the change in the length of one or more of the co-ordinates, and of the positions of the points in which the co-ordinates cut the planes.

5. The cause by which a body is set in motion, whatever be its nature, is called a Force. Forces may be of various descriptions, but as they all produce motion, they may be compared with each other, and made commensurable by means of the motion they produce; and we judge of the intensity of a force by the quantity of motion it is capable of causing. As we know nothing of forces, except by their effects, we may hence assume that the force is always measured by the quantity of motion it impresses upon a point, and the latter is always proportioned to the velocity of this point.

6. For the more convenient comparison of forces, we measure them in terms of some conventional force, taken as the unit. Of this we have a practical illustration in the manner in which the forces of steam-engines and water-wheels are compared, in terms of the conventional force called a horse-power. The intensity of forces estimated in terms of some conventional unit, may then be denoted by numbers, expressed algebraically by letters, or represented by lines of definite magnitude.

7. The circumstances which must be known in respect to a force, besides its intensity, are—the place where it acts, or, its point of application and its direction. The point of application is defined in the mode we have already explained, by referring it by means of co-ordinates, to three rectangular planes. The direction of a force is that in which it tends to cause a point to move: it is usually represented by a straight line drawn in that direction from the point of application; and if upon this line be set off the number of units from a scale, which corresponds to the measure in terms of the conventional force, used as the means of comparison, the force will be represented by it, both in magnitude and direction. The direction will be defined in respect to the three co-ordinates of the point of application, by means of the three angles which it makes with these three lines.

In order to give the method all the extension of which it is capable, these angles must be estimated of all magnitudes, from 0° to 180°. The co-ordinates, by this method, need not be conceived to be produced beyond the point of application; and when in calculation we employ the angular functions, those which have different algebraic signs in the first and second quadrants, will

be best suited to express the position in which the line of direction lies. Thus when the cosine is used, and its algebraic sign is positive, the line will lie on the side of the point of application towards the plane to which the co-ordinate is perpendicular; and when it is negative, it will be turned towards the opposite direction.

Between the cosines of the three angles the direction of a force makes with the co-ordinates of its point of application, there is a constant relation which may be thus expressed:

$$\cos.^2 a + \cos.^2 b + \cos.^2 c = 1;$$

for the line of direction will be the diagonal of a right-angled parallelopipedon, whose sides are in the direction of the three co-ordinates, and the parts cut off from the latter will respectively represent the cosines of the angles they make with the line of direction, the latter being the radius; now as the square of the diagonal of a right angled parallelopiped is equal to the sum of the squares of its three sides, the square of radius, or unity, is equal to the sum of the squares of the three cosines.

If the line of direction lies in the same plane with the two co-ordinates, with which it makes the angles a and b, the angle c becomes a right angle, and its cosine $=0$, hence, in this case,

$$\cos.^2 a + \cos.^2 b = 1.$$

When all the forces that are under consideration are parallel to each other, one of the axes may be so taken as to be parallel to their direction; two of the angles in this case become right angles, and the equation becomes

$$\cos.^2 a = 1.$$

8. When more than a single force acts upon a body, it is obvious that it will not move in a direction, or with an intensity that is due to one of the forces alone, but will be influenced by all the forces collectively. Hence, when a number of forces act upon the same body, they respectively modify each other, and may under certain circumstances completely neutralize each other. When forces thus destroy each other's action, equilibrium is said to exist among them, and the body on which they act is said to be in equilibrio, under their joint action. It has been found most easy to deduce the expressions which denote the motion of a body, from those which denote the conditions of equilibrium; hence it becomes necessary that the conditions, under which forces produce equilibrium, should be first investigated.

CHAPTER II.

Equilibrium of Forces acting in the same line.

9. The simplest case of equilibrium is when two forces act in the same line, with equal intensities, but in contrary directions. We represent this contrariety of direction by means of the Algebraic signs + and —. When more than two forces act in the same line, it is obvious that equilibrium can only exist when the joint intensities of those that act in one direction, are exactly equal to the joint intensities of those which act in opposition to them. Expressing this difference of direction, by considering one set of forces as negative in respect to the other, we obtain the algebraic expression

$$A+B+C+\&c.=0,$$

or in words.

Equilibrium exists among a number of forces acting in the same straight line when their sum is equal to 0.

10. When a number of forces acting upon a body are not in equilibrio, we may, without altering the circumstances under which the body is placed, conceive their united action to be replaced by that of a single force, under which the body would move exactly, as if the whole continued to act. A force which thus produces the same effect as a number of others, and may therefore identically replace them, is called their Resultant; the several forces whose action it thus identically replaces are called its Components.

11. If a force equal in magnitude to the Resultant, but contrary in direction, be applied to the point at which the latter would act if its components were removed, it will be obviously in exact equilibrium with the components; for the case becomes that of two equal forces acting in the same straight line, but in contrary directions. Hence if any number of forces be in equilibrio, any one of them must be equal in magnitude, and contrary in direction to the resultant of all the rest. If, therefore, we have the relations that exist between the Resultant and its components, we can thence deduce the conditions of equilibrium of any forces whatsoever.

These relations may for convenience of investigation be divided into three cases, those of:

1. Forces converging to a point;
2. Parallel forces;
3. Forces acting in one plane, but neither parallel nor converging to a single point.

In considering the latter case, we shall have occasion to speak of the conditions of equilibrium of forces acting in any direction whatsoever, but all our applications can be referred to the case of their being situated in one plane.

CHAPTER III.

Equilibrium of Forces converging to a point. Composition and Resolution of Forces.

12. The Resultant of the two forces converging to a point, is represented both in magnitude and direction, by the diagonal of a parallelogram constructed on the two forces as sides.

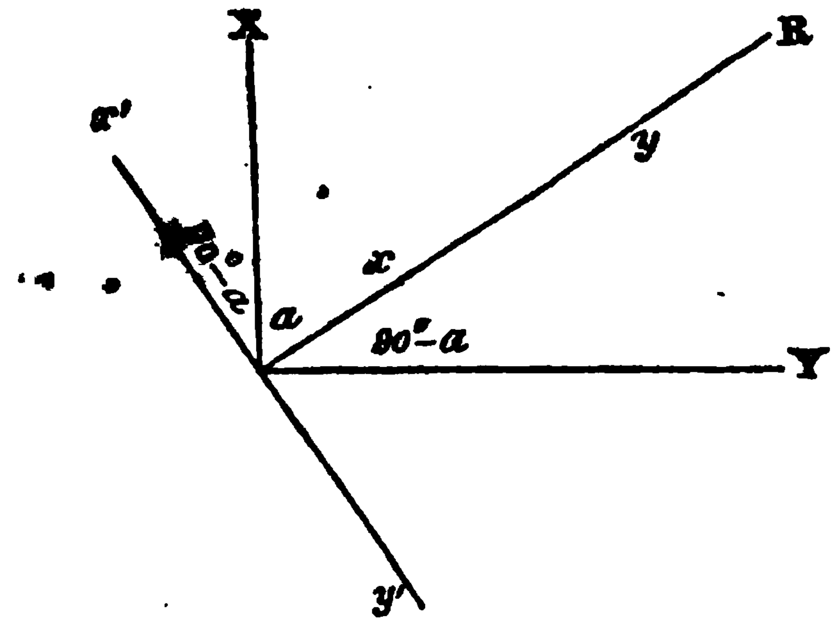
First:—Let the directions of the two forces be at right angles to each other, and call the one X, and the other Y. Let R be the unknown magnitude of the resultant, and a the angle which it makes with the direction of X. If we suppose the two forces to be extremely small and represented by their differentials dX and dY, and that they vary according to the same law, so that when dX becomes successively $2d$X, $3d$X, &c. dY becomes $2d$Y, $3d$Y, &c.; it will be evident that the angle a will not vary, and that the resultant will be constant in its direction; its increase will also follow the same law with its components, and if represented at first by dR, it will become in similar succession $2d$R, $3d$R, &c. Thus in the successive increments of the three forces, the ratios between the resultant R and the two components X and Y will remain constant. The relation $\frac{X}{R}$ being constant, may be represented by a function of the constant angle a. This fact may be expressed by the notation

$$\frac{X}{R}=\varphi(a). \tag{1}$$

But as the angle comprised between Y and R is also constant, the relation between these two quantities may be represented by some function of it; and this function will obviously be of the same form with that which represents the former relation, or to express it algebraically, the angle being the complement of a

$$\frac{Y}{R}=\varphi(90^\circ-a). \tag{2}$$

In order to render the rest of the investigation more obvious, we must have recourse to the annexed figure.



In this, the rectangular forces X and Y are represented in magnitude and direction with the undetermined resultant R lying between them, for its direction must of necessity be intermediate, and in the same plane with them. Let us now consider the force X as the resultant of two others, the first of which x is in the direction of the force R, and the second x' is at right angles to it; the angle comprised between the directions of X and x is the same with that contained between X and R, or is equal to a, and the angle contained between X and x' is $90°-a$. The same relation will then exist between these three forces, taken by pairs, that exists between X, Y, and R, or

$$\frac{x}{X}=\varphi(a), \quad \frac{x'}{X}=\varphi(90°-a). \tag{3}$$

But from the equations (1) and (2) we have, multiplying by R,

$$X=R.\varphi(a), \quad Y=R.\varphi(90°-a), \tag{4}$$

and from the equations (3) we obtain in like manner

$$x=X.\varphi(a), \quad x'=X.\varphi(90°-a), \tag{5}$$

substituting the values of $\varphi(a)$, and $\varphi(90°-a)$ from equations (1) and (2), we obtain

$$x=\frac{Y^2}{R}, \quad x'=\frac{XY}{R}.$$

Resolving Y in a similar manner into the two rectangular components y and y', one of which y is in the direction of R, we obtain by a similar operation

$$y=\frac{Y^2}{R}, \quad y'=\frac{XY}{R}.$$

The force R being the resultant of the two, X and Y, is also the resultant of their four components x, y, x', y', whose values are

$$\frac{X^2}{R}, \frac{Y^2}{R}, \frac{XY}{R}, \frac{XY}{R},$$

but the two last x' and y', are equal in magnitude, and because they respectively make right angles with R on its opposite sides; they act in the same line in contrary directions; hence they mutually destroy each other's actions, and the resultant R is made up of the two remaining forces, which act in the same direction with it, or

$$R=\frac{X^2}{R}+\frac{Y^2}{R};$$

whence

$$R=\sqrt{(X^2+Y^2)}, \qquad (6)$$

which is the expression for the magnitude of the diagonal of a right angled parallelogram whose sides are X and Y; therefore the resultant of two rectangular forces is represented in magnitude by the diagonal of the parallelogram constructed on the two forces as sides.

That it is also represented by it in direction will be obvious from a few simple considerations.

The value of R being thus determined, call the angle which the diagonal of the parallelogram makes with the side X, b, then

$$X=R. \cos. b; \qquad (7)$$

substituting the value of X from the equation (4) and dividing by R

$$\cos. b=\varphi.(a).$$

The unknown function of the angle a may therefore be always represented by the cosine of the known angle b; and if there be any case in which $a=b$, the equality must hold good in all others. Now if the forces X and Y be equal, the resultant R must be equidistant in direction from the directions of the two forces, and the angle a will become the angle which the diagonal of the parallelogram makes with the side X, or $a=b$; therefore the resultant of two rectangular forces is not only represented in magnitude, but in direction, by the diagonal of the parallelogram constructed upon the two forces as sides.

Next suppose that the two forces, X and Y, are not rectangular, but make with each other any other angle a. Resolve X in two other forces, one of which, x, is in the direction of Y, the other x' perpendicular thereto; the resultant will therefore be the resultant of the three forces Y, x, and x', but as Y and x are in the same direction, they have a resultant which is equal to their sum, and R becomes the resultant of two rectangular forces, whose magnitudes are respectively $Y+x$, and x', and from (6)

$$R=\sqrt{[(Y+x)^2+x'^2]};$$

or

$$R^2=(Y+x)^2+x'^2=Y^2+2\,Yx+x^2+x'^2, \tag{8}$$

but

$$X^2=x^2+x'^2, \text{ and } x=X \cos.\ a,$$

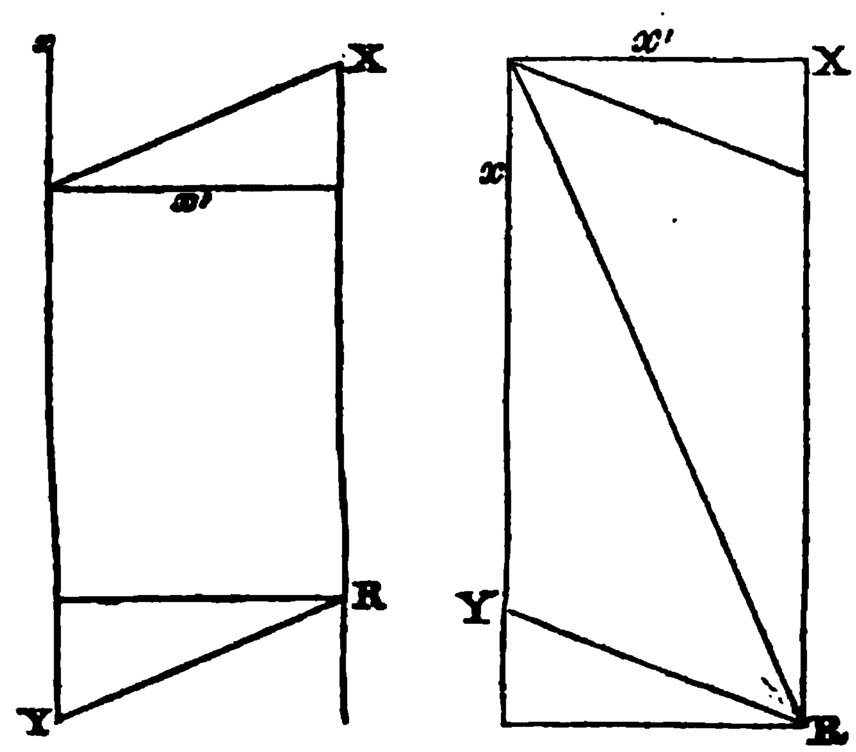
substituting these values in the equation, (8),

$$R^2=X^2+2XY \cos.\ a+Y^2, \tag{9}$$

which shows that the resultant R is represented in magnitude by the diagonal of the parallelogram constructed upon the forces X and Y as sides. Also is it represented in direction, for the diagonal of the parallelogram constructed on $Y+x$ and x' as sides, is identical not only in magnitude, but in position with the diagonal of the parallelogram constructed on X and Y.

The angle, a, may vary between 0° and 180°; when it exceeds a right angle, the quantity, x, becomes negative in respect to Y; the quantity, cos. a, also becomes negative, and the second term of the equation is negative.

In order to render this part of the investigation more clear, we annex a construction, in the two cases of obtuse and acute angles, contained between the two forces.



13. The problems that have relation to finding the resultant of two forces, when the components are given, and to finding the components when the resultant is given, go by the name of "The Composition and Resolution of Forces." All the questions in which forces are to be composed or resolved, may be solved by means of plane trigonometry; and in general, of two forces, their resultant, and the three angles they respectively make with each other, any two being given, the remainder may be found.

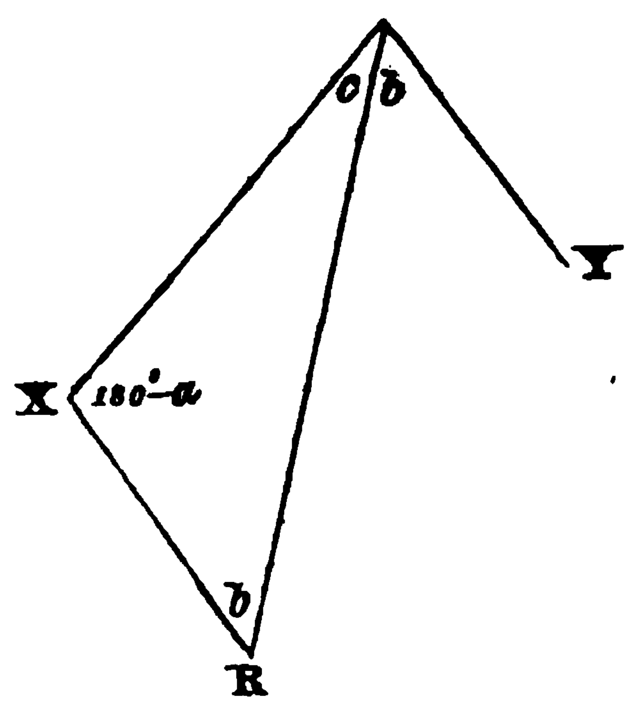
(1). When the two components and the angle they contain are given, we have from the equation, (9),

$$R=\sqrt{(X^2+2XY \cos.\ a+Y^2)}; \tag{10}$$

when the angle, a, is a right angle,

$$\left.\begin{array}{c} R=\sqrt{(X^2+Y^2)}, \\ X=R\ \cos.\ a,\ Y=R\ \sin.\ a. \end{array}\right\} \qquad (11)$$

(2). In the figure beneath, the force X, the resultant R, and a line equal and parallel to Y, make up a triangle, of which the angles are: the supplement of a; the angle b, equal to the angle contained by the force Y and the resultant R; and the angle c, contained by the resultant and the force X; hence as the sides of triangles are proportioned to the sines of the opposite angles,



$$R : X : Y : : \sin.\ a : \sin.\ b : \sin.\ c, \qquad (12)$$

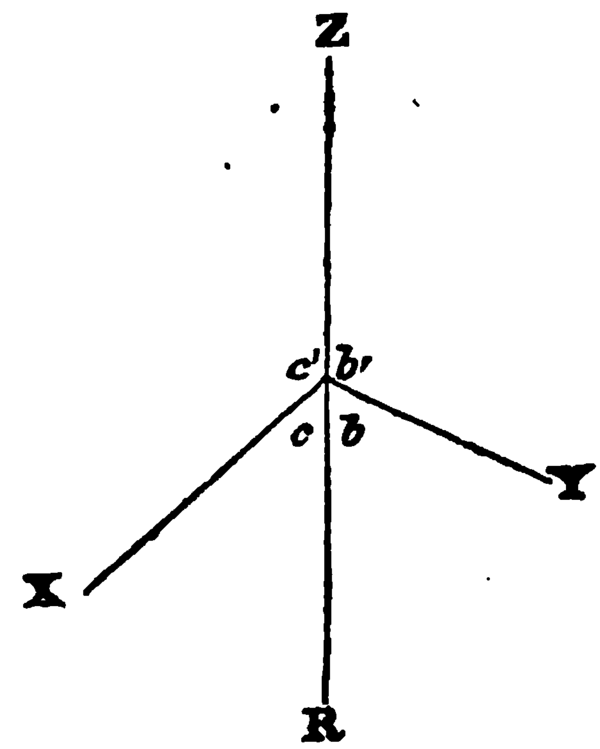
and if the directions of X and Y are rectangular,

$$R=\frac{X}{\cos.\ c}=\frac{Y}{\cos.\ b}, \qquad (13)$$

$$X=R\ \cos.\ c,\quad Y=R\ \cos.\ b. \qquad (14)$$

14. Three forces converging to a point are in equilibrio when each is proportioned to the sine of the angle contained by the directions of the other two; they are also in equilibrio when represented by the three sides of a plane triangle; and hence the directions of the three forces must lie in the same plane, and the sum of any two must be greater than the third.

Let the three forces be X, Y and Z, (by § 10), the force Z



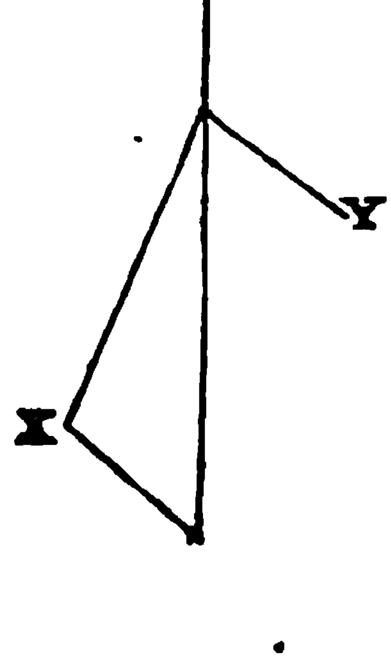
will be equal to the resultant of X and Y and contrary in direction, hence the relation between their magnitudes may be expressed, by substituting Z for R in analogy (12), as follows:

$$Z : X : Y : : \sin. a : \sin. b : \sin. c; \qquad (15)$$

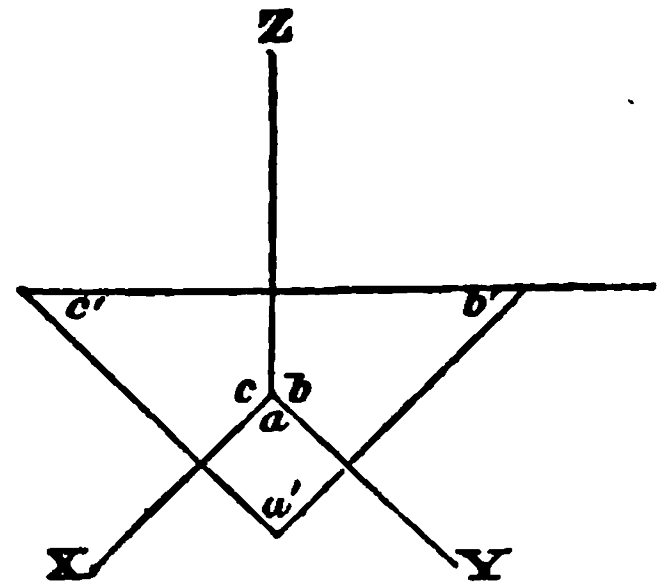
but the angle a, is the angle contained by the forces X and Y; the angle b, is the supplement of the angle b' contained by the directions of Z and Y; and the angle c is the supplement of the angle c' contained between the directions of Z and X; and as the sines of angles and their supplements have the same magnitude,

$$Z : X : Y : : \sin. a : \sin. b' : \sin. c'; \qquad (16)$$

or the forces are each proportioned to the sines of the angles contained by the directions of the other two. As the forces X and Y, with their resultant R, are represented in magnitude by the three sides of a triangle, so, as Z=R, the three forces X, Y and Z are also represented in magnitude by the sides of a triangle. This triangle may be formed by drawing, through the extremity of the line representing one of the forces X, a line equal and parallel to Y, and joining the ends of the last line to the point at which the forces act. This last line is obviously equal to Z, and in the same direction produced, for it is the diagonal of the parallelogram, of which X and Y are sides.



A triangle whose sides are proportioned to the forces, and perpendicular to their direction, may be formed as in the figure beneath, by drawing perpendiculars from points taken at will in the



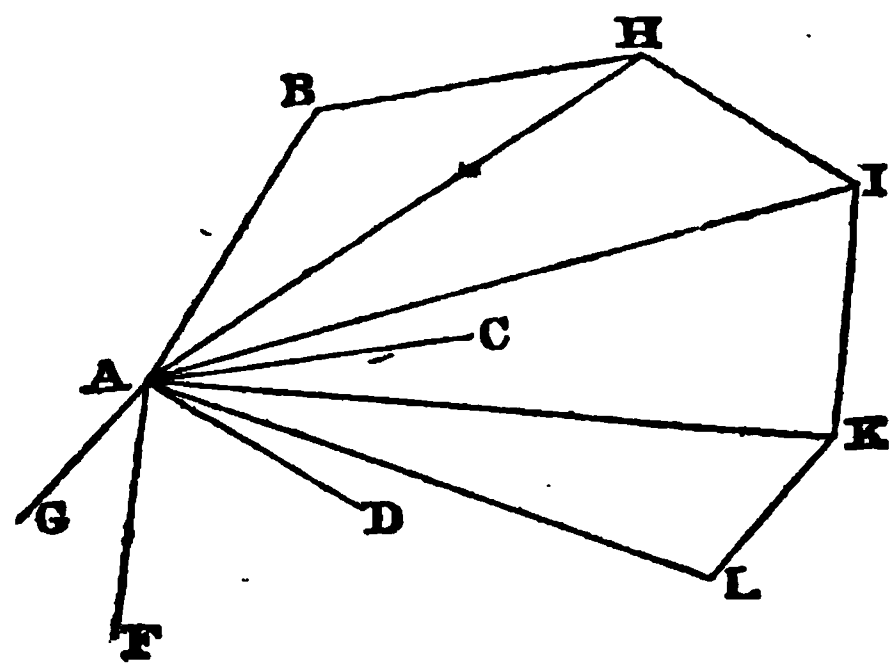
directions of the forces. It will be manifest that the three angles a', b', c', of this triangle are respectively the supplements of the three angles a, b, c, that the direction, of the forces make with each other; hence the triangle thus constructed will be similar to that formed in the former construction, whose sides represent the magnitude of the three forces.

Three oblique forces cannot be in equilibrio, unless their directions converge to a single point, for the resultant of any two of them must be equal and opposite to the direction of the third; and hence its direction must pass through the point to which the directions of the others converge.

As the resultant of any two of the forces lies in the same plane with them, being the diagonal of a parallelogram of which they are the sides, the third force, which is in the direction of this resultant produced, must also lie in the same plane. The three forces that are in equilibrio being represented in magnitude by the three sides of a triangle, the sum of any two of which must be greater than the third, the same must be true of the sum of the magnitudes of any two of the forces.

15. When we have it in our power to find the resultant of any two forces, we may proceed to find the resultant of three or more; for as the resultant identically replaces its components, we may, after finding the resultant of any two of the forces, conceive them to be removed, and the resultant substituted; the resultant of this and the third foree, will be the resultant of the three forces; and this last resultant may be again combined with a fourth force, and so on.

This problem may be illustrated by a remarkable geometric construction.



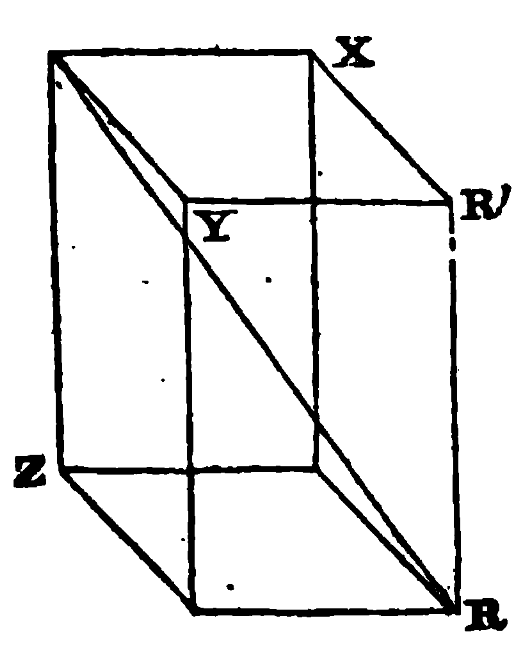
Let AB, AC, AD, AF, AG, represent a number of forces concurring to a point; through the point B, draw the line BH, equal, and parallel to AC; through the point H, draw HI, equal and parallel to AD; through the point I, draw IK, equal and parallel to AF; and through the point R, draw KL, equal and parallel to AG; then the line which joins L to A; will be the common resultant.

It is obvious that the line AH is the resultant of AB and AC; AI the resultant of AH and AD, or of AB, AC, and AD; the line AK, of AI and AF, or of AB, AC, AD and AF; and the line AL, of AI and AG, or of all the forces AB, AC, AD, AF, and AG.

This construction is called the polygon of forces. If the polygon close, or the last line, drawn parallel to the last force, end at the point A, the forces are in equilibrio, and the resultant is equal to 0.

16. The resultant of three rectangular forces, converging to a point, is represented in magnitude and in direction, by the diagonal of the rectangular parallelopiped, whose sides represent the three forces in magnitude and direction.

The resultant R′, of two of the forces X and Y, will be the diagonal of the rectangle which forms one of the faces of the parallelopiped, of which these forces are sides, and



$$R' = \sqrt{(X^2+Y^2)};$$

the resultant R, of R', and Z will be the diagonal of the rectangle of which these two forces are sides, and which will cut the parallelopiped into two equal and similar prisms; it will therefore be the diagonal of the parallelopiped, and will be represented by the formula

$$R = \sqrt{(R'^2+Z^2)} = \sqrt{(X^2+Y^2+Z^2)}. \qquad (17)$$

If we call the angle the direction of R makes with X, a; the angle the direction of R makes with Y, b; the angle the direction of R makes with Z, c; these angles are, as has been shown, connected by the following relation,

$$\cos.^2 a + \cos.^2 b + \cos.^2 c = 1.$$

We may find the values of X, Y and Z, when the force R, and the angles a, b, and c, are given, by the resolution of three plane triangles, of each of which the hypothenuse and an angle are given, thus:

$$X = R \cos. a, \quad Y = R \cos. b, \quad Z = R \cos. c. \qquad (18)$$

17. In those investigations in mechanics where a number of forces are concerned, it is usual to resolve them all into three forces parallel to the three co-ordinates; and the resultant of these three sets of rectangular forces will obviously be the common resultant of all the forces. The formulæ given above (18) furnish a convenient mode of effecting this.

Call the several forces F, F', F'', &c. the angles they respectively make with the co-ordinates of their points of application a, b, c, &c., a', b', c', &c., a'', b'', c'', &c. the values of X, Y and Z, in (18) become

$$F \cos. a, \quad F \cos. b, \quad F \cos. c, \text{ \&c.}$$

and calling the three rectangular forces, which are the sum of the components of all the original forces X, Y and Z,

$$\left.\begin{array}{l} X = F \cos. a + F' \cos. b + F'' \cos. c + \text{\&c.} \\ Y = F \cos. a' + F' \cos. b' + F'' \cos. c' + \text{\&c.} \\ Z = F \cos. a'' + F' \cos. b'' + F'' \cos. c'' + \text{\&c.} \end{array}\right\} \qquad (19)$$

In these expressions, X, Y, and Z, are the components of the resultant of all the forces resolved into three, mutually at right angles to each other. The formulæ show that the resultant of any number of forces, resolved into a component in any given direction, is equal to the sum of the components of all the forces, resolved into directions parallel to the general resultant.

18. The three forces, X, Y, and Z, are not situated in the same plane, and hence can never be in equilibrio, so long as any one of them has any magnitude; for three oblique forces, in order to be in equilibrio, must have their directions in one plane; hence the condition of equilibrium among any number of forces, each

resolved into three at right angles to each other, and parallel to the axes of the co-ordinates, becomes

$$\left.\begin{array}{l} F\cos. a+F'\cos. b+F''\cos. c+\&c.=0 \\ F\cos. a'+F'\cos. b'+F''\cos. c'+\&c.=0 \\ F\cos. a''+F'\cos. b''+F''\cos. c''+\&c.=0 \end{array}\right\} \quad (20).$$

19. There is another case in which forces may be in equilibrio, when they converge to a point; this happens when they press the point against a surface, which opposes a resistance to the motion of the point sufficient to prevent its penetrating under the action of the forces. This resistance will be exerted in a direction which is perpendicular, or a normal, to the surface; for were it exerted in an inclined direction, it might be resolved into two components, one of which is parallel, the other perpendicular to the surface; the former would cause the point to move along the surface, while the latter alone would oppose the progress of the point; now as mere resistance can never generate motion, however it may in other respects affect it, the resistance of the surface must be exerted in the direction of the normal alone.

In order then that the point be in equilibrio, it is no longer necessary that the resultant of the forces shall be equal to 0, but merely that its direction shall be a normal to the surface; this it must be, otherwise it would not be opposite in direction to the resistance of the surface, and therefore would not be in equilibrio with it. The action of the surface, being just sufficient to keep the point in equilibrio, may be represented by a force equal to the resultant of all the other forces, but acting in a contrary direction, or by —R; and the condition of equilibrium, they being resolved into three, parallel to their co-ordinates, will be (17)

$$\sqrt{(X^2+Y^2+Z^2)}-R=0, \quad (21)$$

while the pressure they exert upon the surface will be

$$\sqrt{(X^2+Y^2+Z^2)}. \quad (22)$$

CHAPTER IV.

IV. Equilibrium of Parallel Forces. Centre of Parallel Forces.

20. The conditions of equilibrium among parallel forces, may be reached by steps similar to those employed in determining the condition of equilibrium among converging forces. The method of finding the resultant of two of them must be first investigated; a force equal and opposite to this, will be in equilibrio with the two first. We may then proceed to the resultant of three, four, or any number of forces; and a force equal, and opposite to it, will cause equilibrium in the system, and the relations that exist among them will show the condition of equilibrium.

Parallel forces cannot act upon a single point; we therefore suppose the points on which they act, to be connected in some manner, and first, by an inflexible and inextensible line, which is called their line of application.

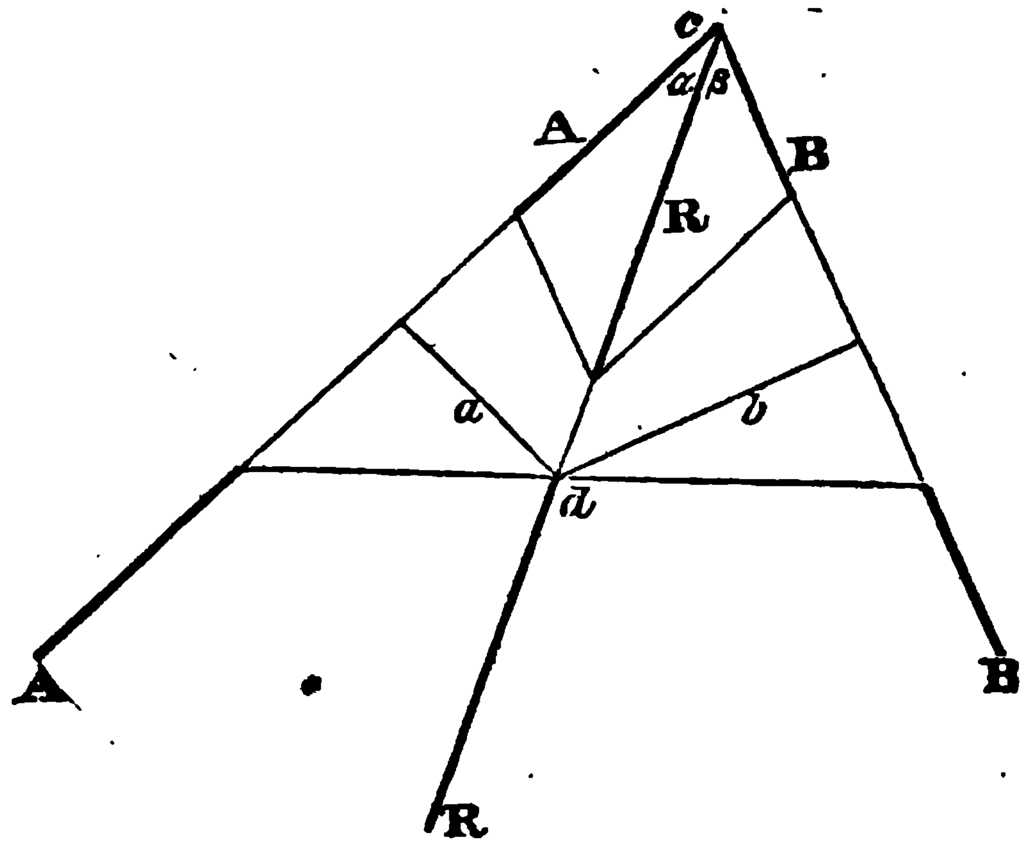
There is one case in which parallel forces have no resultant. This happens when there are but two that act on opposite sides of the line of application, and are equal in magnitude; or when the resultant of all those that act on one side of that line, is exactly equal, but not directly opposite to the resultant of all those that act on the other side. Two forces thus constituted, are called a Couple. It will be obvious that their effect would be to cause the line of application to revolve, and were it free, to move around; the two forces would finally act in the same line, and in opposite directions; they would then cease to be parallel. Two such forces then, so long as they continue parallel, have no resultant, neither can they be in equilibrio.

21. In investigating the method of finding the resultant of parallel forces, it will be seen that we shall consider the point of application of a force to be changed. A force will obviously produce the same effect, whether it act at one or another point of the same inflexible straight line. Of this we have a physical illustration in the fact, that when a simple pressure is exerted through the intervention of a rigid bar, it is wholly unimportant whether the bar be long or short, provided its weight have no influence. We may therefore conceive the point of application of a force to be transferred to any other point in its direction, provided we imagine the two points to be connected by an inflexible straight line.

22. The resultant of two parallel forces is equal to their sum,

parallel to their directions, and divides their line of application into parts, reciprocally proportioned to the magnitude of the two forces.

Suppose first, that, as in the figure, two converging forces act upon the same side of the line, call them A and B; their action



will not be changed by supposing their respective points of application to be transferred to the point C, in which their directions meet; and the direction of their resultant must therefore pass through this point. From the point of convergence, then, its point of application may be transferred to the point d, in which its direction cuts the line of application, and this will therefore be the point of application of the resultant. Call the angle at which the forces are inclined to each other ϱ, the value of the resultant from (9) is

$$R = \sqrt{A^2 + AB \cos.\ \varrho + B^2}\,; \qquad (23)$$

the relation of the forces A and B is from (12)

$$A : B :: \sin.\ \beta : \sin.\ \alpha, \qquad (24)$$

β being the angle the direction of B makes with that of R, and α the angle that the direction of A makes with that of R; but the ratio of the sines will be the same as that of the perpendiculars a and b, let fall upon the respective directions of the forces, from the point of application of the resultant: hence

$$A : B :: b : a. \qquad (25)$$

We shall have occasion hereafter to recur to this step.

Now the equation (23), and the analogy (25) being true, whatever be the magnitude of the several angles the forces and their resultant make with each other, will be true when the lines are parallel. But when the lines are parallel, the angle $\varrho = 180^\circ$ or $= 0^\circ$ and

$$\cos.\ \varrho = 1\,;$$

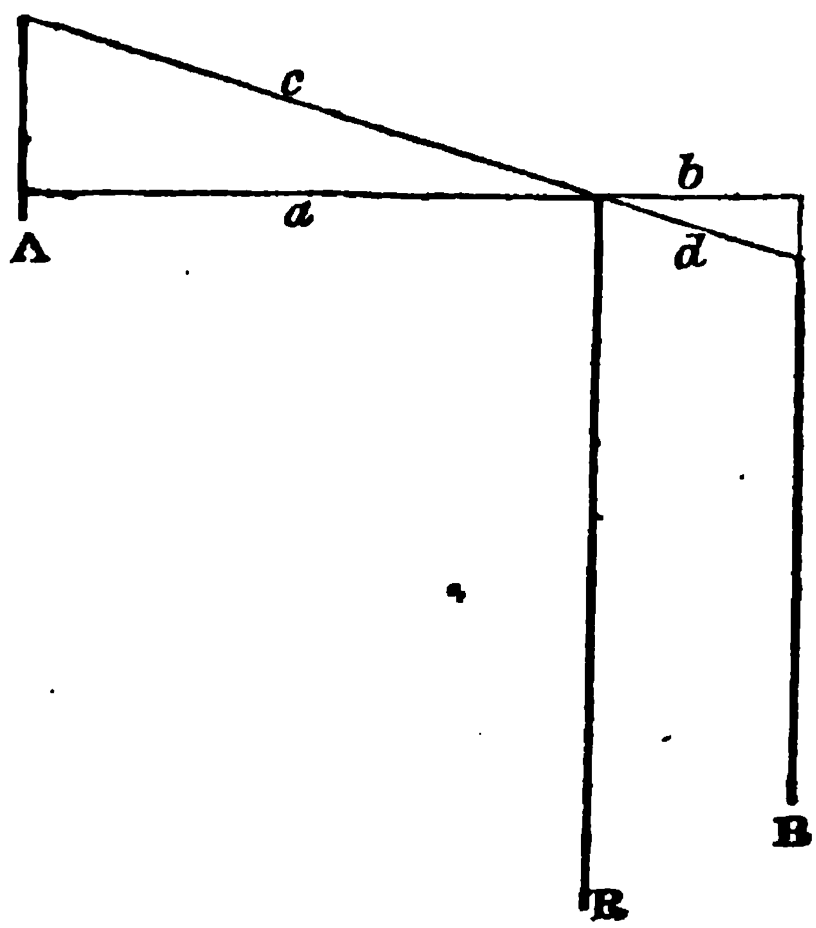
hence the equation (28) becomes

$$R=\sqrt{(A^2+A\,B+B^2)} \text{ and}$$

$$R=A+B \tag{26}$$

or the resultant is equal to the sum of the forces.

The point of convergence being removed to an infinite distance, the direction of the resultant becomes parallel to the directions of the two forces; and



the two perpendiculars a and b become proportioned to the parts c and d into which the point of application of the resultant divides the line of application of the forces, which is hence divided into parts inversely proportioned to the two forces.

If the two parallel forces act on opposite sides of their line of application, the same is also true; but in this case, the opposition of their direction is pointed out, by one of the forces being considered negative in respect to the other; hence their algebraic sum becomes their arithmetic difference. The point of application of the resultant, will fall in the prolongation of the line of application, beyond the point to which the greater force is applied; and the parts into which the line of application is divided, will be measured from the point of application of the resultant to those of the two components.

To make this evident, suppose that to a line on which the two forces, A and B are applied, a third force, C, is also applied equal to the resultant, and opposite in direction; this force will be negative in respect to the others, and will keep the system in equilibrio; the conditions of which will be thus expressed:

$$A+B+C=0, \tag{27}$$

any one of these forces then, is in equilibrio with the other two; and the resultant of these two would be equal to it, and opposite

in direction. Let the two forces, whose resultant is required, be B and C; the resultant being equal and opposite to A, will have a value, which, considering that one of the two forces B and C, is negative in respect to the other, may be thus expressed:

$$R=B+C,$$

and as the point of application of C divides the line to which A and B are applied, into parts inversely proportioned to the two forces, we have

$$A : B :: c : d,$$

and

$$A+B : B :: c+d : d;$$

but

$$A+B=C,$$

and $c+d$ is the whole length of the line of application, measured from the point to which A, or its equal and opposite force, R is applied; hence this line is again divided into parts inversely proportioned to the magnitudes of the two forces.

23. The resultant of any number of parallel forces, acting upon points invariably connected with each other, may be found upon the same principle by which the resultant of any number of converging forces is found. Any two of them may be first resolved into a single one; this may be combined with a third, and the resultant found; the second resultant may be combined with a fourth, and so on. In this way it will be found, that the resultant of any number of parallel forces is equal to their sum.

Call the forces A, B, C, D, &c.

the first resultant R′ being equal to the sum of A and B,

$$R'=A+B,$$

the value of the second resultant is

$$R''=R'+C=A+B+C,$$

and of the last resultant—

$$R=A+B+C+D+\&c. \qquad (28)$$

In this expression, as in the others, the forces that act in opposite directions, must be considered as positive and negative in respect to each other. The condition of equilibrium is obviously

$$A+B+C+\&c.=0. \qquad (29)$$

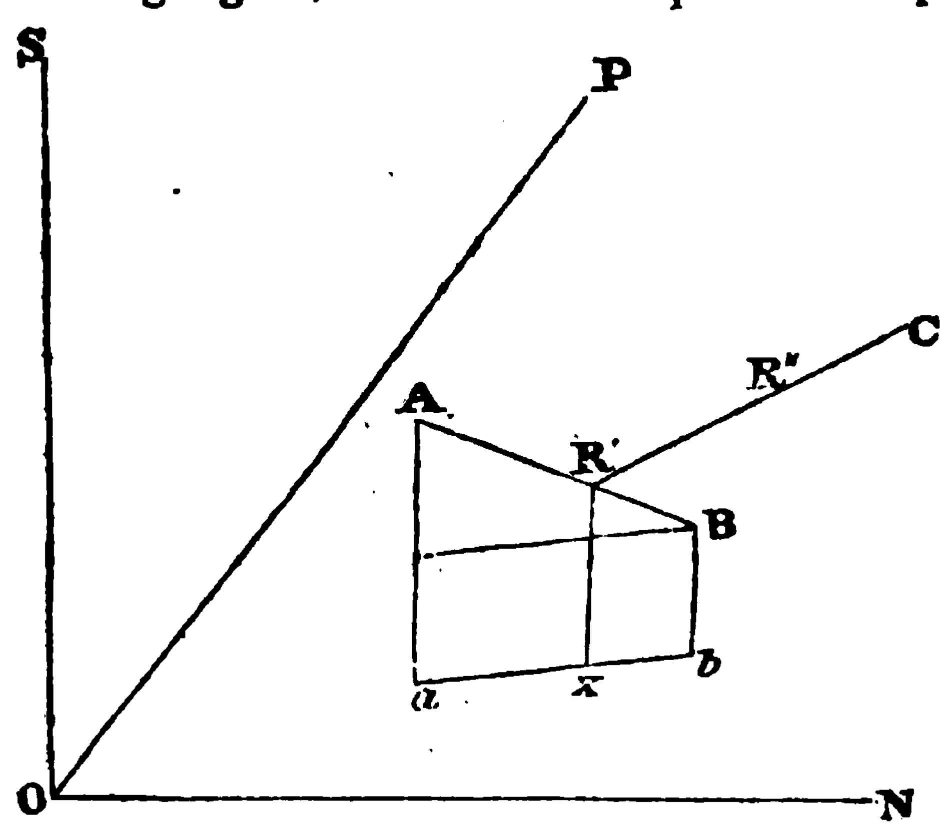
24. A change in the direction of two parallel forces, provided they do not cease to be parallel, produces no change in the position of the point to which their resultant is applied; but the resultant itself will have its direction changed, continuing always parallel to its component. And in the several steps by which the resultant of a number of parallel forces is found, it is obvious that

the position of the several successive, and finally, of the last resultant, remains constant, however the forces vary in direction, provided they do not cease to be parallel. Hence, in a system of parallel forces, if the several forces revolve around their respective points of application without ceasing to be parallel, the resultant will also revolve around its point of application, always retaining its parallelism to its components. From this property, the point of application of the resultant of a system of parallel forces, is called their *Centre.*

25. It is important in many practical cases, to be able to determine the position of the centre of a given system of parallel forces, applied to points forming an invariable, or rigid system. This is effected by finding the value of its co-ordinates, in terms of the several forces, and their respective co-ordinates.

Call the several forces A, B, C, &c., their respective co-ordinates a, a', a'', b, b', b'', c, c', c'' &c., the resultant R, and its co-ordinates X, Y, and Z.

In the following figure, let A and B represent the points of



application of two of the forces A and B, and R′ the point of application of their resultant R′. Let O be the origin of the co-ordinates O P, O N, O S, and a, b, and x the co-ordinates of the two forces, and their resultant, in respect to the plane of O P and O N. The line that unites the points in which the co-ordinates cut this plane, is a straight line, for the three forces are in the same plane. Then as the line A B is divided by the resultant into parts inversely proportioned to A and B

$$A : B :: R'A : R'B;$$

and

$$A+B : B :: AB : R'B,$$

or

$$R' : B :: AB : R'B. \qquad (30)$$

Draw a straight line through the point B, parallel to the line that joins the points in which a, b, and x cut the plane of O P and O N. The respective distances of the points A and R′ from this line will be $a-b$, and $x-b$.

From the similarity of triangles

$$AB : R'B :: a-b : x-b\ ;$$

comparing this with the analogy (30)

$$R' : B :: a-b : x-b\ ;$$

whence

$$R'(x-b)=B(a-b)\ ;$$

now the resultant R′ is equal to the sum of the two forces A and B, and multiplying these equals by b we have

$$R'=Ab+Bb\ ;$$

adding the two last equations

$$R'x=Aa+Bb. \tag{31}$$

A similar chain of reasoning, in respect to each of the other two sets of co-ordinates, gives

$$\left.\begin{array}{l} R'=Aa'+Bb' \\ R'z=Aa''+Bb'' \end{array}\right\}\ ; \tag{32}$$

by division

$$\left.\begin{array}{l} x=\dfrac{Aa+Bb}{R'} \\ y=\dfrac{Aa'+Bb'}{R'} \\ z=\dfrac{Aa''+Bb''}{R'} \end{array}\right\}. \tag{33}$$

Let now C be the point of application of the third force C; R″ the point of application of the resultant of the two forces C and R′, of which the latter is the resultant of A and B. A similar investigation gives for the values of the co-ordinates x' y' z' of the force R″

$$x'=\frac{R'x+Cc}{R'+C}$$

$$y'=\frac{R'y+Cc'}{R+C}$$

$$z'=\frac{R'z+Cc''}{R+C}\ ;$$

substituting the values of $R'x$, $R'y$ and $R'z$,

$$x'=\frac{Aa+Bb+Cc}{A+B+C}$$

$$y'=\frac{Aa'+Bb'+Cc'}{A+B+C}$$

$$z'=\frac{Aa''+Bb''+Cc''}{A+B+C}.$$

It is now obvious that the same method may be extended to any number of forces whatsoever, and that we should finally obtain for the values of the co-ordinates of any number of forces the following equations:

$$\left.\begin{aligned} X&=\frac{Aa+Bb+Cc+\&c.}{A+B+C+\&c.} \\ Y&=\frac{Aa'+Bb'+C'+\&c.}{A+B+C+\&c.} \\ Z&=\frac{Aa''+Bb''+Cc''+\&c.}{A+B+C+\&c.} \end{aligned}\right\} \quad (34)$$

In the case of equilibrium these equations become

$$\frac{Aa+Bb+Cc+\&c.}{A+B+C+\&c.}=0$$

$$\frac{Aa'+Bb'+Cc'+\&c.}{A+B+C+\&c.}=0$$

$$\frac{Aa''+Bb''+Cc''+\&c.}{A+B+C+\&c.}=0$$

The expressions (34) may be made to assume the following form, which is more convenient in its application to the cases that may occur in practice:

$$\left.\begin{aligned} X&=\frac{\Sigma\,.\,Fx}{\Sigma\,.\,F} \\ Y&=\frac{\Sigma\,.\,Fy}{\Sigma\,.\,F} \\ Z&=\frac{\Sigma\,.\,Fz}{\Sigma\,.\,F} \end{aligned}\right\} \quad (35)$$

The numerator of the three fractions being the sum of the products of the respective forces into their distances from the several planes, and $\Sigma\,.\,F$ being the sum of the forces themselves.

26. The most useful applications of these formulæ, are to the determination of the centres of parallel forces in lines, surfaces, and solids. In these cases, the several magnitudes are supposed to be divided into an infinite number of small parts or elements, each of which is acted upon by an equal parallel force. The manner in which the foregoing formulæ are transformed into others adapted to this research, together with a few of their applications will now be given. It will however be obvious, that these formulæ, having reference to the small elements of which the magnitudes are made up, must be differential; and that the discovery of the position of the centre of parallel forces, can only be effected by means of the integral calculus.

In the case of irregular solids and lines of double curvature, it is necessary to refer the position of their elements to three rectangular planes; and hence the three equations for the co-ordinates of the centre of parallel forces, are necessary. In the case of surfaces and plane curves, no more than two of these equations are necessary; for the surface itself, or the plane in which the line lies, may be taken as the plane passing through two of the axes, and it will only be necessary to define the position in respect to these two axes.

In lines that are symmetric on each side of one of their points, in surfaces that are symmetric on each side of a line that traverses them, and in solids of revolution, but one of the equations is necessary; for one of the axes may in the first instance be supposed to pass through the point on each side of which the line is symmetric, and to be perpendicular to the line at that point; in the second it may be assumed as coinciding with the line that divides the surface into two equal parts; and in the third, it may be taken as coinciding with the axis, by a revolution around which the solid is described. In all these cases, the centre of parallel forces will be situated somewhere in the common intersection of the two planes, its distance from which is therefore $=0$, or if these be the planes on which Y and Z fall,

$$Y=0,\ Z=0.$$

Restricting ourselves to cases that admit of the use of no more than one equation (1). In symmetric curves, the equation (35)

$$X=\frac{\Sigma\,.\,Fx}{\Sigma\,.\,F}$$

becomes

$$X=\frac{\int xdl}{l}, \qquad (36)$$

in which l is the length of the line, and x the variable ordinate of the element dl, in respect to the assumed plane.

(2) In plane surfaces the equation becomes

$$X=\frac{\int xydx}{s}, \qquad (37)$$

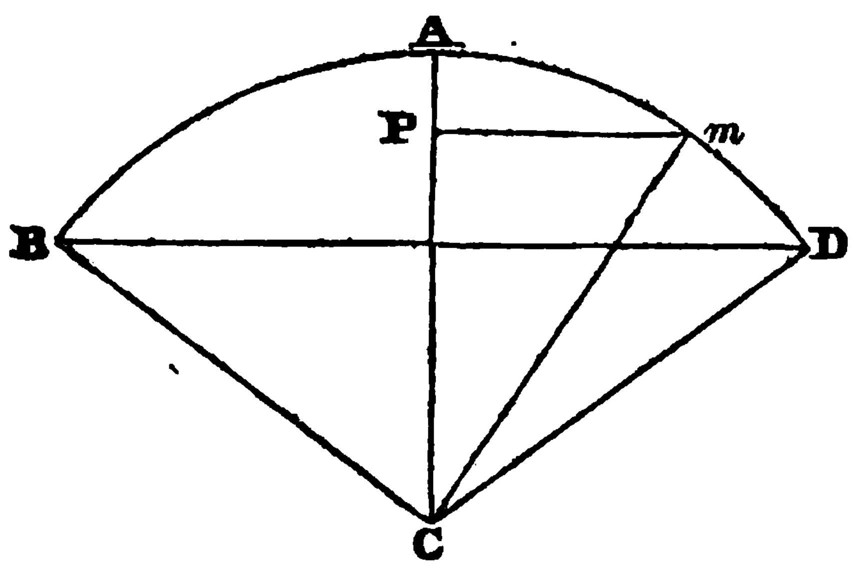
where s is the area of the surface, whose element is ydx.

(3) In solids of revolution, the formula for the position of their centre of parallel forces becomes, π being the ratio of the circumference of a circle to its diameter,

$$X=\frac{\int 2\pi xy^2dx}{\int 2\pi y^2dx}=\frac{\int xy^2dx}{\int y^2dx} \qquad (38)$$

EXAMPLES.

(1) *To find the centre of Parallel Forces of a circular arc.* Let B A D be the circular arc, whose length $=l$.



Place the origin of the co-ordinates at the centre; make the radius C A$=r$, CP$=x$, mP$=y$, A$m=s$;
the differential equation of the arc s is

$$ds=\sqrt{(dx^2+dy^2)},$$

and

$$x=r.\cos.\frac{s}{r^2},$$

whence

$$\int x ds=\int x\sqrt{(dx^2+dy^2)}=r.^2\sin.\frac{s}{r}+\text{C}.$$

When the arc Am becomes equal to AD

$s=\frac{l}{2}$, and the arbitrary constant C$=0$;

and from the equation (36)

$$l\text{X}=\int x dl,$$

therefore

$$l\text{X}=2r^2\sin.\frac{l}{2r}.$$

Let c represent the chord of the arc, then

$$c=2r.\sin.\frac{l}{2r};$$

whence

$$l\text{X}=rc,$$

and

$$l:c::r:\text{X};$$

wherefore the distance of the centre of gravity of a circular arc, from the centre of the circle, is a fourth proportional to the length of the arc, the chord, and the radius.

By analogous processes, the centres of parallel forces may be found in other curves, that are symmetric on each side of the point in which the axis cuts them, thus:

The centre of gravity of an arc of a cycloid, that is divided into two equal parts by the diameter of the generating circle, is at one third of the perpendicular height of the arc, from the vertex.

In a semicircle $c=2r$, and $l=\pi r$ hence

$$\pi r : 2r :: r : \mathrm{X};$$

$$\mathrm{X}=\frac{2r}{3.1416}=.63662r.$$

(2) *To find the centre of parallel forces in a segment of a circle.*

In the same figure let the radius $\mathrm{CA}=r$ and let the part of AC intercepted between the centre and the chord $\mathrm{BD}=a$; let the centre of the circle again be the origin of the co-ordinates; the equation of the circle will be

$$r^2=x^2+y^2;$$

whence $$y=\sqrt{(r^2-x^2)};$$

the expression (37) gives

$$s\mathrm{X}=2\int\sqrt{(x^2-y^2)}xdx.$$

Integrating between the values $x=a$, and $x=r$

$$s\mathrm{X}=\frac{2}{3}(r^2-a^2)^{\frac{3}{2}},$$

and the chord $$c=2\sqrt{(r^2-a^2)};$$

whence

$$\mathrm{X}=\frac{1}{12}\cdot\frac{c^3}{s}.$$

Applying analogous methods, we obtain the following results:

A triangle has its centre of parallel forces, in the line drawn from its vertex to the point, that bisects its base, at two-thirds of its length from the vertex.

In a trapezium, two of whose sides are parallel, call these two sides c and d, and the straight line which bisects both, a, and let the origin of the co-ordinates be in the point where this line cuts the side c;

$$\mathrm{X}=\frac{a}{2}\cdot\frac{c+3d}{c+d}.$$

A sector of a circle, has its centre of parallel forces in the radius that divides it into two equal parts; and its distance from the centre of the circle is a fourth proportional to the arc, the chord, and two thirds of the radius.

The distance of the centre of parallel forces of a parabola, from its vertex, is equal to three fifths of its axis.

The centre of parallel forces in a cycloid, is in the diameter of the generating circle, at the distance of one fourth of that line from the vertex.

In the surfaces that bound solids of revolution, π being the ratio of the circumference of a circle to its diameter, the element of the surface is $2\pi ydx$, hence the equation (38) becomes

$$X=\frac{\int 2\pi xydx}{\int 2\pi ydx}=\frac{\int xydx}{s};$$

which being identical with (37,) shows, that the distance of the centre of parallel forces, in the surface formed by the revolution of a plane curve around an axis, is as far from the origin of the co-ordinates, as the centre of parallel forces of the curve by whose revolution it is generated.

(3) *To find the centre of parallel forces, of a solid generated by the revolution of an arc of an ellipse, or of the segment of a spheroid.*

The formula (38) is

$$X=\frac{\int xy^2dx}{\int y^2dx};$$

the equation of the curve is

$$y^2=\frac{c^2}{a^2}+(ax-x^2),$$

a being the fixed axis, and c the revolving axis of the spheroid; whence

$$X=\frac{\int(ax-x^2)xdx}{\int(ax-x^2)dx}.$$

Integrating, and taking the vertex for the origin of the co-ordinates

$$X=\frac{\frac{1}{3}ax^3-\frac{1}{4}x^4}{\frac{1}{2}ax^2-\frac{1}{3}x^3}=\frac{4a-3x}{6a-4x}x.$$

When the segment is a hemispheroid $x=\frac{1}{2}a$, and $2x=a$, which being substituted for a,

$$X=\frac{5}{8}x;$$

its centre of parallel forces is therefore in the fixed axis, at a distance of $\frac{5}{8}$ths of its length from the vertex.

These expressions being independent of the value of c, are true also of spheric segments.

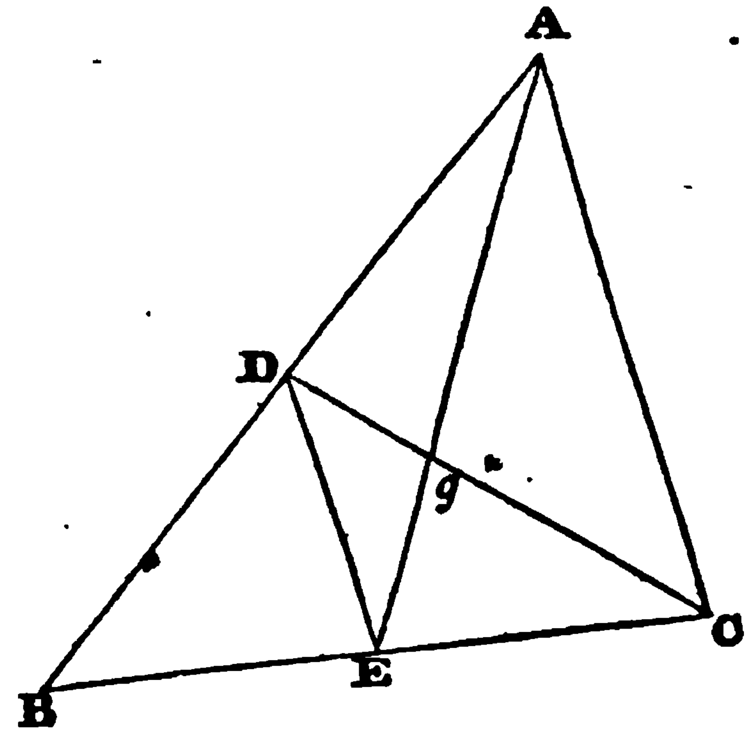
In a hyperboloid of revolution

$$X=\frac{4a+3x}{6a+4x}.$$

In a solid paraboloid, the centre of parallel forces is at the distance of two-thirds of the length of the axis, from the vertex.

Geometric methods also may be applied to the discovery of the position of the centre of parallel forces. Thus in the case of a triangle, it may be shown geometrically, as it has been analytically, to be in the line that joins the vertex to the point which bisects the base, at the distance of two thirds of its length from the vertex.

In the triangle ABC, bisect the base BC in E, the side AB in D, draw AE, CD, and join DE.



The surface being divided into two equal parts by the line AE, the centre of parallel forces will lie in this line; for the same reason it lies in the line CD, and must therefore be in the point g where they intersect each other.

By the similarity of triangles,

$$gD : gC :: gE : gA,$$
$$gE : gA :: DE : AC,$$
$$DE : AC :: AB : DB,$$
$$gE : gA :: AB . DB;$$

but because the line AB is bisected in D

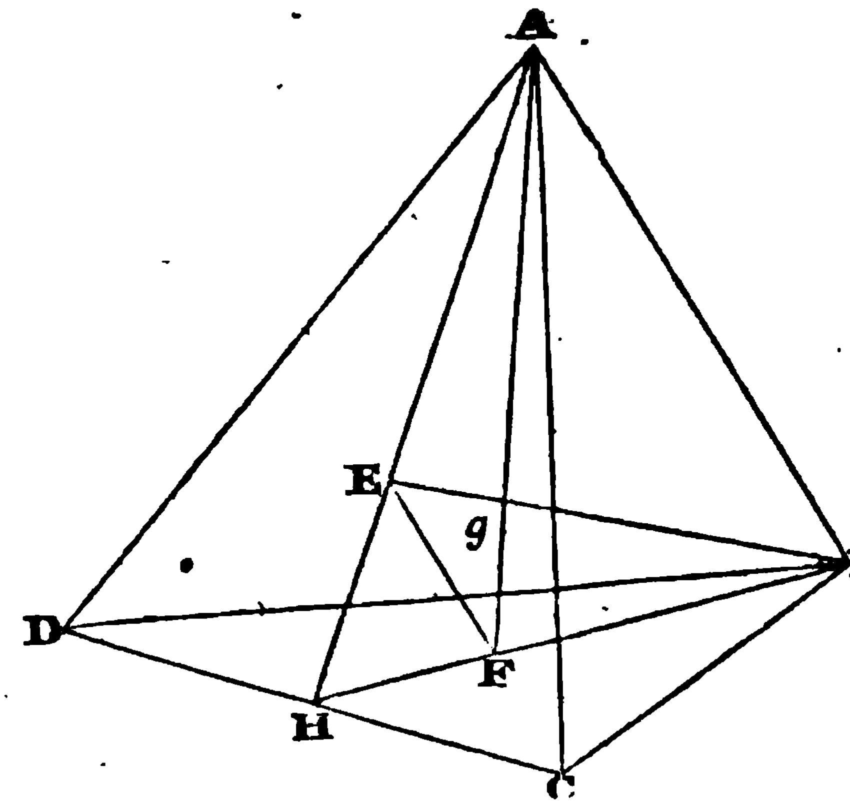
$$AB : DB :: 1 : 2,$$
$$gE : gA :: 1 : 2,$$
$$AE : gA :: 3 : 2,$$

therefore

$$gA = \tfrac{2}{3}AE.$$

In a triangular pyramid the centre of parallel forces is in the line that joins the vertex to the centre of parallel forces of the base, at the distance of three fourths of that line from the vertex.

Let ABCD be a triangular pyramid, to the point H that bisects



the line DC, draw AH, BH; make HE$=\frac{1}{3}$ AH, HF$=\frac{1}{3}$ BH; E and F will be respectively the centres of parallel forces of the surfaces ADC, DBC; join EF, AF, BE. If the pyramid be resolved into elements, by means of planes parallel to DCB, it is manifest that the line AF must pass through the centres of parallel forces of all the elements; their common centre of parallel forces must therefore be in that line; for the same reason it is the line BF; it is therefore at their common intersection, in the point g.

By similar triangles

$$\begin{aligned} Ag : gF &:: AB : FE :: AH : EH, \\ AE : EH &:: 3 : 1, \\ AH : EH &:: 4 : 1, \\ Ag : gF &:: 3 : 1, \\ Ag : AF &:: 3 : 4; \end{aligned}$$

whence $Ag=\frac{3}{4}AF$.

The first of these propositions may be applied geometrically to find the centre of parallel forces of any polygon whatsoever; for it may be divided into a number of triangles, equal to the number of sides less two. Let the centres of parallel forces be found in two of the triangles, join them by a straight line, and divide it into parts inversely proportioned to the areas of the two triangles, the point of division is evidently their common centre of parallel forces. If this point be joined to the centre of parallel forces of the third triangle, and divided in a similar manner, the point of division becomes the common centre of parallel forces of the three triangles. This method being continued, until all the triangles into which the polygon has been divided have been used, the point last obtained is the centre of parallel forces of the whole surface.

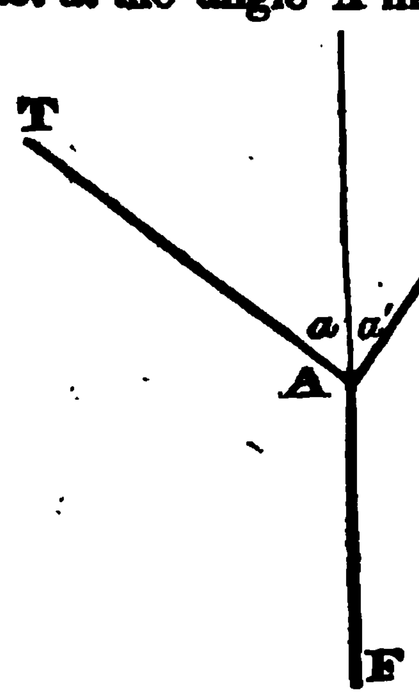
A solid bounded by plane surfaces may be divided by planes, into a number of triangular pyramids. The second of these propositions therefore gives us a geometric method, founded on the

same principles, of finding the centre of parallel forces in any solid whose boundaries are plane surfaces.

27. When the system of points on which a set of forces, whether parallel or not, acts, are not so connected as to form a rigid and invariable system, the conditions of equilibrium are different. They may be investigated in a general manner, but the applications being limited in practice to a few cases, and the investigation difficult, we confine ourselves to the cases: of polygons formed of invariable lines, whose angles are capable of changing their magnitudes under the action of the forces that are applied, until the system reaches a state of equilibrium, and when the forces are parallel among themselves; and of curves, formed by the action of an infinite number of forces, upon the points of a flexible but inextensible line.

28. When a system of parallel forces acts upon a system of inflexible and inextensible lines, forced to move around their connecting points, the sum of the forces will be equal to 0; the forces will be all in one plane, in which the lines at whose angles they act are situated; and the forces will be to each other respectively, as the sum of the cotangents of the angles their directions make with the lines, at whose point of concourse they themselves act.

Let a force F act at the angle A made by two inflexible lines



AT, AT', which angle is variable under the action of the forces; the system will be in equilibrio, when the lines are drawn, each in its respective direction, by forces T and T' that are such as would be in equilibrio with F when acting at the point A. This is evident, because we can conceive the point of application of a force removed to any point in its direction, without changing its action, provided the points be, as in the case before us, connected by inflexible and inextensible lines.

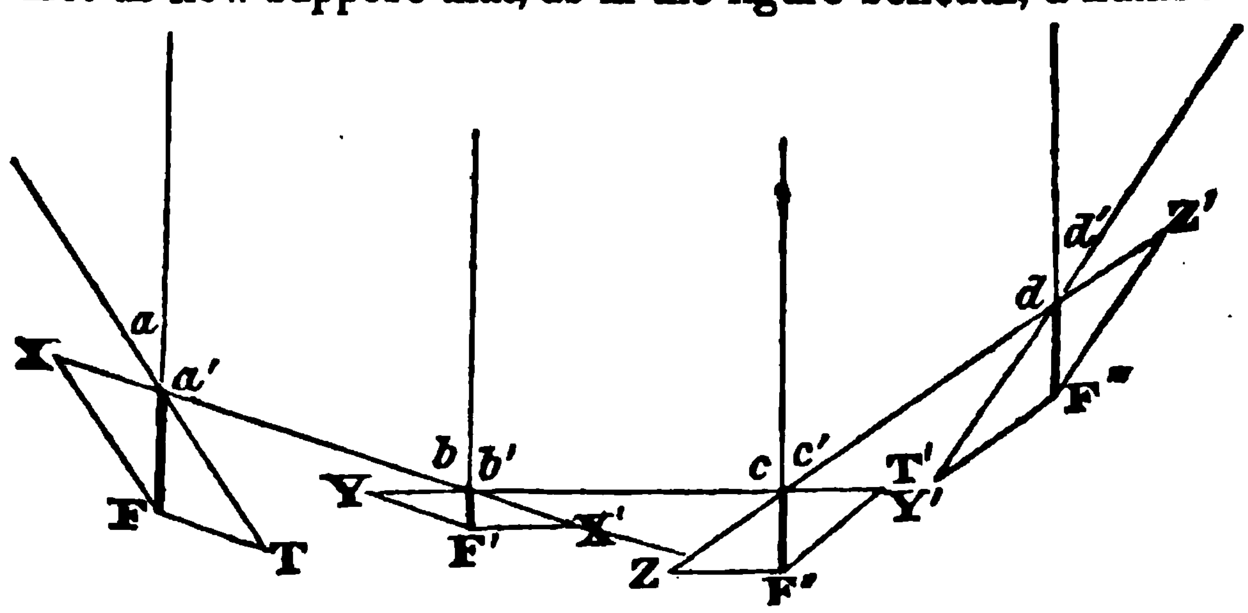
Let a and a' be the angles the direction of the force F makes with T and T'; their sum is the angle these two forces make with each other, hence by (16)

$$F : T : T' : : \sin. a+a' : \sin. a' : \sin. a. \qquad (39)$$

As three forces in equilibrio around a point must be in one plane, the direction of the lines AT, AT', will be in the same plane with the direction of F.

It will be obvious that the more obtuse the angle, made by the directions of the lines becomes, the greater will be the tension they have to support under the action of the force F. If the direction of the force F bisect the angle made by the two lines, each of them will bear an equal tension.

Let us now suppose that, as in the figure beneath, a number of



parallel forces, F, F', F'', F''', act upon a system of rigid and inflexible lines in such a manner as to cause equilibrium; resolve the force F into two others, T and X, in the direction of the two lines at whose angles it acts; the second force into two others, X' and Y, in the direction of the two lines at whose angle it acts; resolve in like manner, F'' into two Y' and Z; and F''' into Z' and T'. These forces will have from (16) the following relations:

$$\left.\begin{array}{l} F : T : X : : \sin. (a+a') : \sin. a' : \sin. a \\ F' : X' : Y : : \sin. (b+b') : \sin. b' : \sin. b \\ F'' : Y' : Z : : \sin. (c+c') : \sin. c' : \sin. c \\ F''' : Z' : T' : : \sin. (d+d') : \sin. d' : \sin. d \end{array}\right\} . \qquad (40)$$

But in order that the polygon, formed by the lines at whose angles the forces act, shall be in equilibrio, the forces by which each of the lines is drawn at its opposite ends, must also be in equilibrio, or

$$X=X', \quad Y=Y', \quad Z=Z'.$$

The several values of these forces, obtained from the analogies, are therefore equal by pairs, or

$$\left.\begin{array}{l} \dfrac{F \sin. a}{\sin. (a+a')}=\dfrac{F' \sin. b'}{\sin. (b+b')}, \\ \dfrac{F' \sin. b}{\sin. (b+b')}=\dfrac{F'' \sin. c'}{\sin. (c+c')}, \\ \dfrac{F''' \sin. c}{\sin. (c+c')}=\dfrac{F \sin. d'}{\sin. (d+d')} \end{array}\right\} ; \qquad (41)$$

but when the forces are parallel sin. a' = sin. b, sin. b' = sin. c, sin. c' = sin. d.

Multiplying the equations just given (41), by one or other of these equals, we have

$$\left.\begin{array}{l}\dfrac{\text{F sin. } a \text{ . sin. } a'}{\text{sin. } (a+a')}=\dfrac{\text{F}' \text{ sin. } b \text{ sin. } b'}{\text{sin. } (b+b')}\\ =\dfrac{\text{F}'' \text{ sin. } c \text{ sin. } c'}{\text{sin. } (c+c')}=\dfrac{\text{F}''' \text{ sin. } d \text{ sin. } d'}{\text{sin. } (d+d')}\ ;\end{array}\right\} \quad (42)$$

but

$$\frac{\text{sin. } a \text{ sin. } a'}{\text{sin. } (a+a')}=\frac{1}{\text{cot. } a+\text{cot. } a'}\ ;$$

and so of the rest. The expressions may therefore take another form, and become

$$\left.\begin{array}{l}\dfrac{\text{F}}{\text{cot. } a+\text{cot. } a'}=\dfrac{\text{F}'}{\text{cot. } b+\text{cot. } b'}\\ =\dfrac{\text{F}''}{\text{cot. } c+\text{cot. } c'}=\dfrac{\text{F}'''}{\text{cot. } d+\text{cot. } d'}\end{array}\right\}\ ; \quad (43)$$

hence each of the forces is proportioned to the sum of the cotangents of the two angles its direction makes with the two lines, at whose junction it acts.

The lines are also in one plane, in which the forces F, F′, F″, &c. likewise act.

The forces F, T and X, being in equilibrio around a point, are in the same plane; in which X′ lies also; and F′ being parallel to F, and drawn from a point in the direction of X is also in the same plane; Y lies in this plane also, because it is in equilibrio with X′ and F′; and for the same reason that F′ was in the same plane with F, F″ lies in the same plane with F′; thus all the forces lie in a single plane, and the polygon is a plane figure.

The forces which act to keep the system in equilibrio, being parallel, their sum is equal to 0; and if the system be attached at the two extremities, to two fixed points, on which the tensions resolved into two parallel directions, are T and T′,

$$\text{F}+\text{F}'+\text{F}''+\&\text{c.}+\text{T}+\text{T}'=0.$$

The sum of the parts of the tensions which act in directions, parallel to those of the forces, will therefore be equal to the resultant of all the other forces.

29. When the polygon has its two extremities fixed, it is called the Funicular Polygon, because it is, as will be hereafter shown, the figure a rope would assume when loaded by weights attached to different points of its length. When the points at which the forces act become infinitely near, the polygon becomes a curve that may be called the funicular curve; and when the weights are equal, the curve is called the Catenaria. The research of the equations of the Catenaria is not strictly an object of elementary me-

chanics, we shall therefore give them without investigation, referring our readers for farther information to the works of Poisson and Venturoli.

Let l denote the length of the curve;
l' the horizontal distance between the points of suspension.
c the angle made by l' with a tangent to the curve at the point of suspension;
A the tension of the chain at the same point;
T the tension at some given point;
x a variable abscissa;
y the corresponding ordinate;
s the corresponding arc;
h the weight of a lineal unit of the length of the chain.

The equations are

$$(1)\quad \frac{l'}{l}=\frac{\cos. c}{\sin. c}\text{ hyp. log. }\frac{\cos c}{1-\sin. c},$$

$$(2)\quad \frac{A \sin. c}{h}=\frac{1}{2}l,\text{ or } A=\frac{hl}{2\sin. c},$$

$$(3)\quad T=\surd(A^2-2Ahs\,.\sin. c+h^2s^2)$$

which at the lowest point becomes

$$T=A\cos. c,$$

$$(4)\quad \frac{1}{2}l=\frac{A\sin. c}{h},\text{ and }\frac{1}{2}l'=\frac{A\cos. c}{h}\text{ hyp. log. }\frac{\cos. c}{1-\sin. c},\qquad (44)$$

$$(5)\quad x=\frac{A\cos. c}{h}\text{ hyp. log. }\left\{\frac{A-hy\mp\surd[(A-hy)^2-A^2\cos.^2 c]}{A(1-\sin. c)}\right\}$$

$$(6)\quad y'=\frac{A(1-\cos. c)}{h},$$

y' being the ordinate of the lowest point of the curve.

$$(7)\quad s=\frac{A\sin. c}{h}\mp\surd\left\{\frac{[(A-h)y^2-A^2\cos.^2 c]}{h}\right\}$$

CHAPTER V.

EQUILIBRIUM OF FORCES IN THE SAME PLANE, BUT NEITHER PARALLEL, NOR CONVERGING TO A SINGLE POINT.

30. We now proceed to consider the equilibrium of forces that act in the same plane, but are neither parallel, nor converge to a single point. Let us suppose them to act upon points, so united as to form an invariable system. The resultant of any two of them may be found by prolonging their directions until they meet, and conceiving their resultant applied to the point of concourse; the direction of this resultant may be produced, until it cut the direction of the third force, and a second resultant calculated as if they all acted there; this may be combined in the same way with a fourth force, and so on, until all the forces have been used. It is obvious then, that generally speaking, such forces will have a resultant. The only case indeed, in which they cannot, will be that in which the resultant of all the forces but one, is equal to the remaining force, and parallel to its direction, but acts upon the opposite side of the line that joins their points of application. Two such forces have no resultant, and as has been seen (§ 20) are called a Couple.

The investigation of the conditions of equilibrium in such forces, may be conducted by resolving each of them into two others, parallel to two rectangular axes, whose values can be obtained by the formula (18). Two systems of parallel forces, are thus obtained at right angles to each other; the resultant of each determined in magnitude, and the position of its points of application found. Equilibrium of course, can only take place when these two resultants are each equal to 0; calling the forces F, F′, F″, &c.; the angles they respectively make with their co-ordinates aa', bb', cc', &c. The conditions of equilibrium become,

$$F\cos. a+F'\cos. b+F''\cos. c+\&c.=0$$
$$F\cos. a'+F'\cos. b'+F''\cos. c'+\&c.=0.$$

The same reasoning will be applicable to forces acting in any manner in space, whose conditions of equilibrium will require a third equation of similar form, to be joined to the two we have just given; the angles whose cosines are used, being those each force makes with its three co-ordinates.

If the forces are not in equilibrio, and have a resultant, call the resultants of the two systems of parallel forces X, and Y; then

$$X=F\cos. a+F'\cos. b+F''\cos. c+\&c.$$
$$Y=F\cos. a'+F'\cos. b'+F''\cos. c'+\&c.$$

and by (17) the resultant R is

$$R=\sqrt{(X^2+Y^2)};$$

call the angles its direction makes with the two axes, at the point, in which if produced, they will meet, α and β. Then by (18),

$$\cos.\ \alpha=\frac{X}{R},\ \cos.\ \beta=\frac{Y}{R}.$$

All that remains is to determine the co-ordinates of the point of application. These may be determined, when the position of the points of application of the several forces are determined, and their co-ordinates given, by means of the principle contained in formulæ (34, and 35). In these, the values of the several forces, with those of their co-ordinates, are to be substituted in a manner too obvious to need description.

31. The value of the resultant of forces acting in one plane, may also be determined by means of what are called their Moments of Rotation.

To understand the meaning of this term, we shall recur to the investigation of the value of the resultant of two parallel forces, (§ 22.) In the course of that, it was found that the perpendicular distances of the directions of two converging forces, acting upon an inflexible line, from the point of application of their resultants, and the forces themselves, were in inverse proportion, or as represented by, (25),

$$A : B :: b : a,$$

hence,

$$Aa=Bb.$$

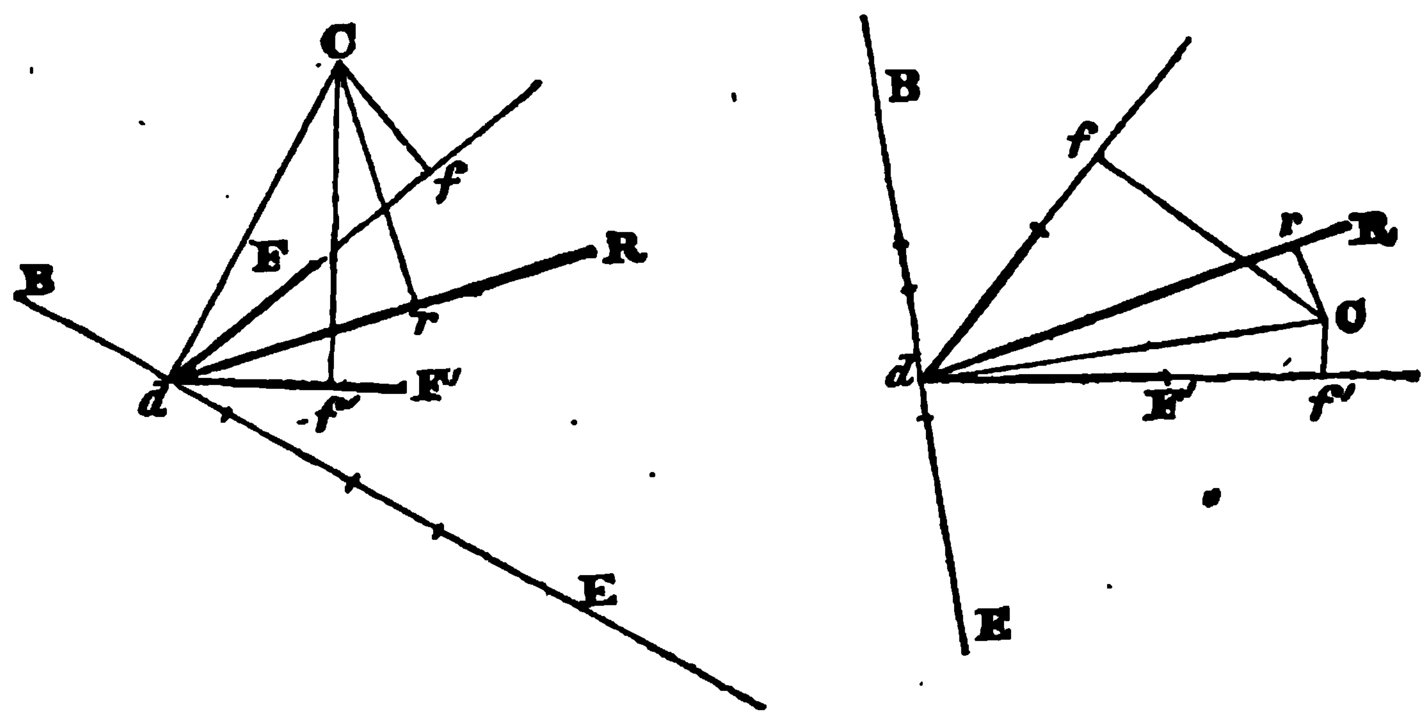
With two forces, having this relation, a third applied to the point of application of the resultant, equal to it, and opposite in direction, would cause the system to be in equilibrio. Let us now suppose that instead of applying a force to this point, it becomes fixed, but in such a manner that the line of application of A and B, may be free to revolve around it as a centre of motion. The two forces although unequal in magnitude, are still in equilibrio, and each will tend to make the line revolve with equal energy. This energy, then, may be expressed by the two equal products Aa, and Bb; and in general, if we suppose the point of application of any force to be united to a fixed point, by an inflexible line, the force will act to cause the line to revolve around that point, with an energy determined by the product of its intensity, into the perpendicular distance of the fixed point, from the direction of the force; hence:

32. The moment of rotation of a force, in respect to a point, is the product of the intensity of the force into the perpendicular let fall upon the direction of the force.

33. The moment of rotation of two forces, in respect to any point situated in their plane, is equal to the sum or difference of

the moments of rotation of its components, in respect to the same point; to the difference, when the point falls within the angle, formed by the directions of the two components; to the sum when the point falls without this angle.

Let F and F′ be the two forces, R their resultant, converging to the point d; C the point whence the perpendiculars are let fall



let f, f', and r, be the three perpendiculars; let the distance C d, $=c$. Let each of the forces F, F′, and R′, be decomposed in two others, one in the direction of c, the other perpendicular to it, or in the direction B d E.

The value of the component of R, in the direction BdE, will be

$$R \cos. \angle RdB;$$

but

$$\cos. \angle RdB = \frac{r}{c};$$

and the component of R, in the direction of B d E, becomes

$$R \cdot \frac{r}{c}.$$

In the same manner, the components of F and F′, in the same direction B d E, may be shown to be

$$F \cdot \frac{r}{c}, \text{and } F' \cdot \frac{r}{c};$$

these will be in the same direction, when the point C falls without the angle, and in opposite directions when it falls within. Now, as these three forces would be in equilibrio, if R were applied in a reversed direction; their components in relation to two rectan-

gular axes, would be equal also; the components of F and F', are therefore equal to the component of R, and

$$R.\frac{r}{c}=F.\frac{f}{c}\mp F'.\frac{f'}{c};$$

multiplying by c

$$Rr=Ff\mp F'f',$$

which expresses our proposition.

Extending the investigation in the usual manner to any number of forces, we have

$$Rr=Ff\mp F'f'\mp F''f''\mp\&c. \tag{45}$$

In this expression, it is obvious that the signs + and —, express the tendency of the force to turn the system, in one or the other direction, around the centre of the Moments.

In case of equilibrium, the expression becomes

$$Ff\mp F'f'\mp F''f''\mp\&c.=0, \tag{46}$$

or ?

34. A system of forces, acting in one plane upon a system of points invariably connected, will be in equilibrio, when the sum of the moments of all the forces that tend to make the system turn in one direction, is equal to the sum of the moments of all the forces that tend to make the system turn in an opposite direction. The same proposition is also true in the case of the system being firmly attached to the point C, or to the centre of the moments; for were the two sets of moments unequal, one or the other would preponderate, and would make the system revolve. In this case it is not necessary that the resultant of the forces should be equal to 0, but merely, that, as the moment of the resultant is equal to the sum or difference of the moments of all the forces,

$$Rr=0.$$

But this can only happen, if R have any magnitude, when

$$r=0;$$

hence the resultant must pass through the fixed point; and,

35. When a system of forces is applied to a system of points invariably connected together in one plane, and having one fixed point, the direction of their resultant must pass through that point, or the system will not be in equilibrio.

36. If we suppose a straight line to be drawn through the centre of the moments, and perpendicular to the plane, this line will become an axis, on which the forces would tend to make the system revolve; and it will be no longer necessary that the forces should act in one plane, but merely that they act in planes parallel to each other; for their moments, determined by lines drawn per-

pendicular both to their own direction and the axis, would remain constant.

37. If the forces do not act in planes perpendicular to the fixed axis, each of them may be resolved into two, one parallel to the axis, the other lying in a plane perpendicular to it. It will then be obvious, that the former produces no effect to make the system revolve; and no more of the force is exerted, for that purpose, than is represented by the latter. The line that represents that component of a given force, which acts in a plane different from that in which it is itself situated, corresponds with its geometric projection in that plane; and calling the force A; the projection P; and the angle the two planes make with each other i; this force may be found (14) by the formula.

$$P = A \cos. i. \qquad (47)$$

38. The conditions of equilibrium in forces acting upon points invariably connected, also hold good, when the points are connected in any manner whatsoever; for it is evident that if the system be in equilibrio, the state of equilibrium will not be changed, by uniting the points of which it is composed in an invariable manner. But in addition to the conditions, that are alone necessary in points connected in an invariable manner, and are common to systems connected in any manner whatsoever, there will be others, that will depend upon the manner in which the points are connected.

BOOK II.

OF MOTION.

CHAPTER I.

Of Motion in general. Uniform Motion. General Principles of Variable Motion.

39. When a material point is acted upon by forces under whose action it is not in equilibrio, it is set in motion, as it also would be by the action of a single force. If it be not acted upon by any force, as there is no reason that it should move in one direction rather than another, it will remain at rest; so also when once set in motion, and no force act, or if the forces that do act are in equilibrio, it must continue to move uniformly forwards in a straight line; for there is now no reason why it should change either the rate, or the direction of its motion. Hence, all bodies will continue in the same state, either of rest, or of motion uniformly forwards in a straight line; unless they be compelled to change their state of rest or of motion, by the action of some force impressed upon them. The truth of this principle is not obtained from abstract reasôning alone, but is the uniform result of observation and experience. Although from what is observed to occur in moving bodies near the surface of the earth, we might at first sight infer that they had a natural tendency to come to rest; still, when we remark, that the more we lessen the resistances, the longer is the continuance of the motion; and that we can in almost all cases, ascribe the diminution of the motion, or its change of direction, to forces that we know from other circumstances to be acting; we infer, that were these resistances not to act, the body would go on uniformly in a straight line. This principle is sometimes ranked as a property of matter, and is called its Inertia. A similar inference may be deduced from the motions of the heavenly bodies. In these, since the earliest record of authentic observation, no change has been detected, that is not pe-

riodic; their mean rates of motion have therefore been constant, although no force has been applied to maintain their motions.

40. In order to represent the circumstances of motion, which consist in the passage of a body through a portion of space in some definite time, we make use of the term Velocity. Velocity is the relation, or ratio, of the spaces described to the times employed in describing them. It will be obvious that in different motions, such a relation does actually exist; the more rapid the motion, the less being the time occupied in describing a given space, and the greater the space described in a given time. But space and time are essentially heterogeneous quantities, and incapable of any direct comparison. We are therefore compelled to resort to a means of comparison, by adopting a method, in which the measure of each of these quantities may be considered as an abstract number. Between such numbers, a ratio capable of being expressed does exist. In order to effect this object, we assume some conventional unit for the space, as the foot, for instance: in like manner we assume a conventional unit, for the measure of the time. In terms of these, a ratio, or relation may be expressed. As the velocity, in uniform motion, increases with the space described, while the time diminishes, the relation between them may be thus expressed:

$$v=\frac{s}{t}, \tag{48}$$

where v is the velocity, s the space, and t the time. The first of these will be denoted in the number of the conventional units of space, described by the body in the unit of time.

From this equation we obtain expressions for the value of the space s, and the time t, as follows:

$$s=v\,t, \qquad t=\frac{s}{v}. \tag{49}$$

41. When two uniform motions are to be compared together; as for instance, when we wish to ascertain the time in which bodies moving, or appearing to move, in the same line with different velocities, shall be at the same point, or shall appear to meet; we may estimate the spaces, from some given point in the line, actually, or apparently described, by the two bodies. Let s represent the distance from the fixed point at the time t; b the distance from the same point, at the instant from which the time t is estimated; then the space becomes $s-b$; and the first equation (48) becomes

$$v=\frac{s-b}{t}; \tag{50}$$

whence,

$$s=vt+b. \tag{51}$$

The time t, may be either positive or negative; when positive, it denotes intervals of time subsequent to the passage of the body through the fixed point; when negative, it denotes intervals prior to the body's reaching that point. So also, may s have positive or negative values, which represent its position in respect to the fixed point. In this manner the equation (51) will point out the position of the body for every possible instant of time, in the line that marks out the direction of its motion.

If there be another body moving, or appearing to move, uniformly in the same line with the first, whose velocity is v'; whose distance at the same time denoted by t, from the fixed point is s'; and the distance from the same point at the instant whence t is estimated, is b', the equation of its motion will be

$$s'=v't+b.' \tag{52}$$

In this equation, besides the same relations of positive and negative, among the quantities, that we have pointed out as applicable to the former, the quantity v' will be negative, when the motion is in a direction contrary to that of the first body.

A comparison between the values of these two equations, will point out the relative position of the two bodies in the direction of their motion. When both are at the same instant, at the same distance from the fixed point, $s=s'$, whence,

$$vt+b=v't+b',$$

which gives for the value of t

$$t=\frac{b'-b}{v-v'}.$$

If this value should be found negative, it denotes, that the bodies meet before the instant whence the time is computed.

To give an instance of the application of these formulæ: suppose two bodies to be moving in the same line, and in the same direction, with velocities in the relation of 1000 : 1.

Then, $v=1000$, $v'=1$.

Let A, represent the body whose velocity is greatest; B, that whose velocity is least.

Let A be situated, at the instant whence the time is estimated, at the fixed point; then,

$$b=0;$$

let B be situated at the same instant, at a distance represented by

$$b'=1000,$$

then,

$$b'-b=1000$$

$$v-v'=\ 999$$

$$t=\frac{b'-b}{v-v'}=\frac{1000}{999}=1\frac{1}{999}=1.001,$$

$$s=vt+b=\frac{vb'}{v-v'}=\frac{1000000}{999}$$

$$=1001\frac{1}{999}=1001,001;$$

hence the body A, will overtake the body B, at the end of one of the units of time, and the $\frac{1}{999}$th part of that unit; and will have described a space equal to 1001 $\frac{1}{999}$ units of space. But, estimated decimally, the fractions become the infinite converging series .001001. The ancients, unaware of the fact, that the sum of such an infinite series, was a finite quantity, reasoned from it, to show that the one body would never overtake the other. The argument employed by them, was called the Achilles, and the conditions were the same as those we have chosen for our example.

The same propositions may be applied to uniform motions, in any lines whatsoever; as for instance to motion in re-entering curves, of which the circle would furnish the most simple instance; and the hour, minute, and second hand of a watch, supposed to be fixed upon the same arbor, would afford an apt illustration.

Supposing them all to set off from the same point, $b-b'$ and $b'-b''$ become equal to the circumference of the circle or π, and for two hands,

$$t=\frac{\pi}{v-v'},$$

for all three hands

$$t=\frac{\pi}{v-v'}+\frac{\pi}{v'-v''};$$

the application of this to calculation, is too obvious to require an example.

42. If a point be impressed at the same time with two uniform motions, it moves in a line which is determined by their joint effect; and as each of these motions is due to a force, the point will move as if it were actuated by a single force, which is the resultant of the two forces; hence, from the principles of § 12, it must move in the diagonal of the parallelogram, constructed upon the two forces as sides. If actuated by any number of motions whatsoever, it will move in the direction of the resultant of the forces that cause these several motions.

Instances of bodies that are actuated at the same time, by more than one motion, are innumerable in practice. All bodies retained upon the surface of others, by means of attraction, friction, or any other force, acquire the motion of the bodies on which they rest.

Thus a body in a carriage, or mounted upon a horse, a person in a vessel, and finally all bodies upon the surface of the earth, have a motion due to that of the body on which they rest. This becomes evident upon a sudden cessation of the motion, and is not instantly communicated, as may be perceived immediately after the beginning of the motion. Thus, when a steam-boat is suddenly set in motion, we feel a tendency to move in a direction apparently opposite; this is due to the inertia, which would leave us in our original position in respect to fixed objects on the shore, were not the motion of the vessel communicated to us; and when the same vessel has her progress suddenly checked, we experience a tendency to move forwards, that remains until again counteracted, in the mode in which the motion was first communicated.

A body thrown from a carriage in rapid motion, by a force acting perpendicular to the direction of the motion, does not fall to the ground opposite to the point where the carriage was, when it was projected, but opposite to the point the carriage has reached, at the time it falls to the ground.

In feats of horsemanship, balls are thrown up vertically, at least so far as the action of the rider influences them, but being impressed by the motion of the horse, they fall again into the hand of the rider; the rider may spring directly upwards from the saddle, and fall again upon it, although the horse be at full speed. The directions in which bodies move, being influenced by all the motions with which they are impressed, the position of bodies moving upon surfaces that are themselves in motion, is the same in relation to points in the moving surface, at given instants of time, as it would have been, had the surface remained at rest. Thus we move from place to place, upon the deck of a vessel, whose motion is not disturbed, with precisely the same effort, that we would perform the same distance upon the land; and were we not aware of the vessel's motion from other circumstances, would, as is done by children, ascribe the motion to the surrounding objects.

The same happens to us, from our situation on the surface of the earth. This is impressed with a rapid motion, not only of revolution, but of translation; yet these being to all intents uniform, we ascribe the change of apparent position that our motion causes in the heavenly bodies, to a proper motion existing in them; and it was ages after these apparent motions had been carefully observed by astronomers, before the true cause of the phenomena was detected.

43. Although it is the tendency of matter, if once set in motion, to move forwards forever with uniform velocity, in the same direc-

tion, this is by no means the most frequent case of motion that occurs in nature. There are in fact two distinct species of forces. (1) Those which having acted for a time upon a body, abandon it and leave it to go forward, so far as they are concerned, in a right line with uniform velocity; these are called projectile forces. (2) Those which act during the whole continuance of the body's motion. Such forces will cause changes in the direction and velocity of the body, and produce what in general terms, are called Variable Motions.

44. As, when a point that has at one instant of time been at rest, is afterwards found in motion, we infer the action of some force to cause that motion; so, when a point in motion, whose velocity and direction have been determined, at some instant of time, is afterwards found moving in a different direction, or with different velocity, we also infer the action of some force to produce this change. This force may either have acted for a greater or less time, and then abandoned the point to itself, or it may have been continually acting. A force of the latter description is called an Accelerating Force, because had it acted upon a point originally at rest, it would have given the point an accelerated velocity.

We judge of the fact of the motion of a point being accelerated, by comparing the spaces described in equal times; if after having described a certain space in the unit of time, it shall be found describing a greater space in an equal time, its motion has evidently been accelerated. So, when after having described a given space in the unit of time, it is afterwards found describing a less space in the same time, its motion is retarded. An accelerating force may produce a retarded motion; for it may act in a direction contrary to the motion originally impressed by some other force upon the body; and indeed most retarded motions are due to the action of forces that may be considered under the general head of Accelerating.

45. The time which a point takes to describe a given space, will obviously depend upon the intensity of the force that causes its motion. Thus a force of double the intensity, will cause a point to describe twice the space in an equal time, and so on. As we know absolutely nothing of the nature of forces, but find their action to be proportioned to the spaces described, we might take the spaces described in equal times as the measure of the forces; but as the velocities are proportioned to the spaces, they are also proportioned to the forces; and therefore in uniform motions, where mere material points are concerned, the velocity and force may mutually serve as each other's measures. All the principles and formulæ that have been applied in the previous book to

the composition of forces, are, therefore, applicable to the composition of velocities.

46. In motions produced by a force that acts continually, the velocities are, as we have seen, variable. It is impossible for us to determine from experience alone, whether such a force acts without interruption, or whether it produces its effect by a succession of impulses, separated by inappreciable intervals of time. Whichever of these modes of action be the true one, the results in both cases will be the same; for if we suppose the velocities to be represented by the ordinates of a curve, whose abscissas represent the times; the motion produced by a succession of impulses, separated by infinitely small intervals of time, would correspond to a polygon of an infinite number of sides; and this would be identical with the curve. We therefore consider all variable motions, as made up of a succession of uniform motions, each continued for a very short space of time; and the results obtained, from investigations founded on this principle, are indentical with those that would occur, were the velocity to be continually varying.

In the equation (48) the time and space s and t, becoming infinitely small, will be represented by their differentials, and

$$v=\frac{ds}{dt},\ ds=vdt,\ dt=\frac{ds}{v}; \tag{53}$$

The intensity of an accelerating force, cannot, like that of a force that produces uniform motion, be measured by the velocity, for that is always varying, even although the force may remain constant; but it may be measured by the momentary variations in the velocity. Let dv be the increase of the velocity during the time dt; then, since the increase in the velocity will be the same as it would have been, had the action of the accelerating force not been interrupted, dv will be equal to the product of the force into its time of action dt, or calling the force f,

$$dv=fdt,$$

and

$$f=\frac{dv}{dt}; \tag{54}$$

then since by (53)

$$v=\frac{ds}{dt}, \tag{55}$$

we have

$$f=\frac{d^2s}{dt^2}. \tag{56}$$

Such are the general equations of variable motion; and they are applicable either to the case of its being accelerated or retarded; but in the latter case, dv is a negative quantity.

CHAPTER II.

Of Rectilineal Motion uniformly accelerated, or uniformly retarded.

47. When the spaces that a point describes in equal times, increase or decrease by equal increments or decrements, the motion is said to be uniformly accelerated or retarded. In such a case the quantities dv, and dt, in equation (54), are obviously constant; hence the accelerating force f, is a constant force.

48. We call the velocity that a body would have, were the accelerating force removed, its Final Velocity; or, as we consider the body to move uniformly, for infinitely small intervals of time, it may be considered at its velocity as the end of the given time, and used without the addition of the word final. The space described from rest, in acquiring that velocity under the action of the accelerating force, is called the Space due to that Velocity; and the velocity acquired, is called the Velocity due to the Space.

49. In the motion of a point uniformly accelerated from a state of rest, the velocities are proportioned to the times; the whole spaces described, are proportioned to the squares of the times; and if the times be represented by the series of natural numbers, the acquired velocities will be represented by the series of even numbers; the whole spaces, by the series of square numbers; and the spaces described in the successive units of time, by the series of odd numbers. The measure of the accelerating force is equal to twice the space described under its action from rest, in the first unit of time; and if the accelerating force be removed, the velocity acquired in a given time by its action, is such as would carry the point in an equal time, with uniform motion, through twice the space it has passed through, in acquiring that velocity.

Suppose the point to have, at the instant the accelerating force begins to act, and in the same direction, a velocity$=a$. It will acquire during each successive unit of time, an additional velocity, which as the force is, in the case under consideration, a constant one, will be a constant increment. This increment, which will be the measure of the accelerating force, we shall call g. The velocity being originally a, will become, at the end of the first unit of time, $a+g$;

at the end of the second unit, $a+2g$;

and at the end of the time t, $a+gt$;

or calling the final velocity v,

$$v=a+gt. \qquad (57)$$

If t vary, and become $t+dt$, the space described s, will vary also, and become $s+ds$: ds then, is the space described in the time dt.

If we suppose that for this small interval of time, the velocity is constant, and equal to v, we have by (53),

$$ds=vdt,$$

substituting the value of v from (57)

$$ds=adt+gtdt.$$

Integrating

$$s=at+\frac{gt^2}{2}+b,$$

b being the distance from some fixed point in the direction of the motion; but when no more than a single point is concerned, b may be taken equal to 0, and the equation becomes

$$s=at+\frac{gt^2}{2}; \tag{58}$$

eliminating t by means of equation (57).

$$s=\frac{v^2-a^2}{2g}. \tag{59}$$

If the body be at rest when the accelerating force begins to act, $a=0$, and the equations (57), (58), (59), become

$$v=gt,\ s=\frac{gt^2}{2},\quad s=\frac{v^2}{2g}; \tag{60}$$

from these equations we readily obtain others, which with them, give the value of the several quantities, each in terms of two of the others, as follows, viz.

$$\left.\begin{aligned} s&=\frac{gt^2}{2}=\frac{v^2}{2g}=\frac{tv}{2};\\ t&=\frac{v}{g}=\sqrt{\frac{2s}{g}}=\frac{2s}{v};\\ v&=\sqrt{2gs}=\frac{2s}{t}=gt;\\ g&=\frac{v}{t}=\frac{2s}{t^2}=\frac{v^2}{2s}. \end{aligned}\right\} \tag{61}$$

If the motion continue only for the unit of time

$$s=\frac{g}{2},\quad g=2s,\quad v=2s.$$

The expression $v=gt$, in which g is a constant quantity, shows us that the velocities acquired are proportioned to the times.

The expression $s=\frac{gt^2}{2}$ in which $\frac{g}{2}$ is constant, shows us

that the whole spaces are proportioned to the squares of the times.

Applying the formula $v=\frac{2s}{t}$ to times taken as the series of natural numbers, and calling the space described in the first unit of time, unity; we have, for the successive values of v,

2, 4, 6, 8, &c.

On the same hypothesis we have for the values of s,

1, 4, 9, 16, &c.

The successive differences of this series represent the spaces described, during each successive unit of time, and are,

1, 3, 5, 7, &c.

The expression $g=2s$, when the motion has continued from rest, for the unit of time, shows that the measure of the accelerating force is equal to twice the space described, from rest, in the first unit of time.

And the expression $v=\frac{2s}{t}$ compared with the equation (48) of uniform motion, $v=\frac{s}{t}$ shows us, that a point, with the velocity acquired by moving from rest under the action of a constant force, would pass through twice the space in an equal time.

50. In the motion of a point that is equably retarded by the action of a constant force, the velocities are as the times that remain until the cessation of the motion; and the spaces that remain to be described, are as the squares of the times estimated in the same manner, and as the squares of the velocities. The point will go, before it loses its motion, through half the space that it would have described, in the remaining time, with uniform velocity.

In our previous investigation if applied to this case, g becomes a negative quantity; for the accelerating force acts in opposition to the original velocity. Hence in retarded motion,

$$\left.\begin{aligned} v&=a-gt, \\ s&=\frac{a^2-v^2}{2g}, \\ t&=at-\frac{gt^2}{2}; \end{aligned}\right\} \qquad (51a)$$

when $v=0$, at which time the constant force will have completely destroyed the initial velocity,

$$t=\frac{a}{g}, \quad s=\frac{a^2}{2g}, \quad g=\frac{a^2}{2s}; \qquad (61b)$$

from which equations, the principles we have stated can be readily determined.

51. If the original direction of the motion be not in the same straight line, in which the accelerating force acts, the point must describe a curve. For it will in the first instant of time describe the diagonal of a parallelogram, whose sides represent the uniform velocity, and the measure of the accelerating force; in this direction it would tend to go forward, were it not again acted upon by the accelerating force; this action would produce a second deflection, and so on; and thus the point would describe a polygon of an infinite number of sides, or a curve. Before then we can determine the nature of the line the body describes, it becomes necessary to investigate the general properties of curvilinear motion.

CHAPTER III.

Of Curvilinear Motion.

52. The forces which concur to produce curvilinear motion are, in general, resolved into three, parallel to three fixed rectangular axes. This may be done, as explained in § 16, in respect to any forces whatsoever. The moving point will describe the resultant of these three forces, which will identically replace all the others that act. In the small elements of the time, it will describe the diagonal of a parallelopiped, of which the three rectangular forces, considered as producing uniform motion for the small interval of time, are the sides. Now if we suppose perpendiculars to be let fall, from the successive positions of the moving point, upon the three axes, the points in which these perpendiculars cut the axes, will move along with them; and the motion of each of them will be such as would be due to a rectangular force, that acts parallel to that axis. If any of the forces cease to act, the motion in the directions parallel to the other axes will not be changed. Each of the three rectangular forces, into which all that act are resolved, is therefore independent of the others, and its action may be determined upon the principles laid down in § 46.

The general equation of variable motion (56) is therefore applicable to this case; and calling the three rectangular forces X, Y and Z, and the spaces in these directions, x, y and z, it will become

$$X=\frac{d^2x}{dt^2}, \quad Y=\frac{d^2y}{dt^2}, \quad Z=\frac{d^2z}{dt^2}. \tag{62}$$

These equations contain all that it is necessary to know in respect to the motion. For 1st, by an integration we obtain the values of the three velocities in these directions; these from (55) are

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt}; \tag{63}$$

from the composition of which, the velocity of the moving body is determined;

2. Another integration gives us the co-ordinates in terms of the time:

3. By eliminating t, we obtain the equations of the curve that the body describes.

In the same case, of the resolution of all the forces into three rectangular forces, we shall have

$$v\,dv=X\,dx+Y\,dy+Z\,dz; \tag{64}$$

for, from the above equations we obtain, by multiplying them respectively by dx, dy, dz, and adding them together,

$$X\,dx+Y\,dy+Z\,dz=\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2};$$

but as the axes are rectangular,

$$dx^2+dy^2+dz^2=ds^2;$$

whence we obtain

$$dxd^2x+dyd^2y+dzd^2z=dsd^2s;$$

therefore

$$Xdx+Ydy+Zdz=\frac{dsd^2s}{dt^2}=\frac{ds}{dt}\,d.\,\frac{ds}{dt}=vdv.$$

If the forces act in one plane, we have need only of the two equations,

$$X=\frac{d^2x}{dt^2},\quad Y=\frac{d^2y}{dt^2}. \tag{65}$$

If but two forces, acting in the same plane, are concerned, it may be simpler to employ them, without resolving them each into three parallel to three rectangular axes; the same principles apply to them as to three rectangular forces, and the equation (56) becomes, calling the forces F and F′,

$$F=\frac{d^2f}{dt^2},\quad F'=\frac{d^2f'}{dt^2}. \tag{66}$$

When no more than two forces act, as the successive diagonals are all in one plane, the curve is a plane curve.

CHAPTER IV.

Of Parabolic Motion.

53. If a point be acted upon by two forces not in the same straight line, by virtue of one of which, it would describe a straight line with uniform velocity, and of the other, a straight line with uniformly accelerated velocity; and if the second force act parallel to itself, during the continuance of the motion; the point will describe a parabola under their joint action. The diameters of this parabola are parallel to the directions of the accelerating force, and its parameter is four times the space due to the velocity communicated by the first of the forces.

Let F′ be the projectile force by whose action the body would go on with uniform rectilineal velocity, and F the accelerating force which begins as soon as the force F′ ceases to act, then from equation (66)

$$Fdt=d.\frac{df}{dt},\quad F'dt=d.\frac{df'}{dt}.$$

As F represents the accelerating force, call

$$F=g;$$

then, F′ representing the force that produces the constant velocity,

$$F'=0;$$

hence

$$gdt=d.\frac{df}{dt},\quad 0=d.\frac{df'}{dt}.$$

Integrating

$$\frac{df}{dt}=gt+A,\quad \frac{df'}{dt}=B,$$

A and B representing the initial velocities in the two directions; but as the accelerating force begins to act at the instant whence the motion is to be computed,

$$A=0.$$

And we may measure B in terms of g, and the space through which the body should pass under its action, in order to acquire the initial velocity; by (61)

$$B=\sqrt{2gs};$$

substituting these values, we have

$$\frac{df}{dt}=gt,\quad \frac{df'}{dt}=\sqrt{2gs},$$

$$df=gtdt,\quad df'=dt\sqrt{2gs}.$$

Integrating anew

$$f=\tfrac{1}{2}gt^2, \quad f'=t\sqrt{2gs}.$$

In which expressions the constant quantities are omitted, for $t=0$, $f=0$, $f'=0$, at the same time;

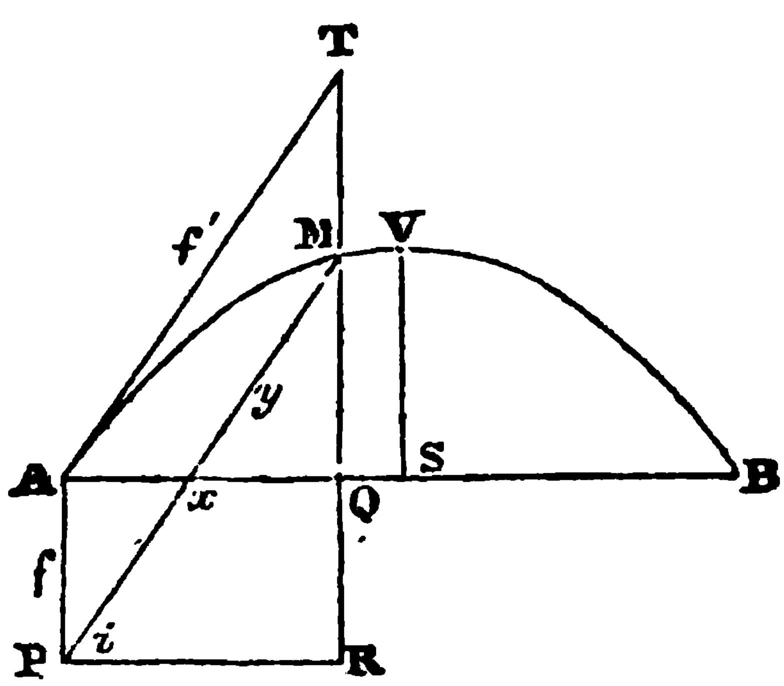
eliminating t

$$f'^2=4sf \tag{67}$$

which is the equation of a parabola, whose ordinate and abscissa are f' and f, and whose parameter is $4s$. f being the abscissa, shows that the diameters of the parabola, and of course its axis, are parallel to the direction of the accelerating force.

To investigate the path by means of two rectangular co-ordinates, X and Y, the directions of X being parallel to the direction of the accelerating force.

In the figure beneath, let AP represent the action of the accele-



rating force, AT the direction in which the projectile force would cause the body to move with uniform velocity.

Draw AQ to represent the direction of the axis X; TR parallel to AP; and PR parallel to AQ; join PM, then

$$AQ=x, \quad QM=y, \quad MR=y+f;$$

call the $\angle TAQ=\angle MPR$, i,

then $\qquad y+f=f' \sin. i, \quad x=f' \cos. i;$

whence

$$y=\frac{x \sin. i}{\cos. i}-f=x \tan. i-f;$$

but from (67)

$$f=\frac{f'^2}{4s}=\frac{x^2}{4s \cos.^2 i};$$

therefore

$$y=x \tan. i-\frac{x^2}{4s \cos.^2 i}. \tag{68}$$

This last equation may be obtained directly, from the resolution of the two forces into two rectangular forces, one of which is parallel to the direction of the accelerating force.

Call the measure of the accelerating force g; as it acts in a direction opposite to the co-ordinate Y, it will be negative, or $-g$.

The alteration in the direction of X, being wholly due to the force that produces a constant velocity, is $=0$.

The two equations (62) become

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt}=g.$$

Integrating,

$$y=-\frac{gt^2}{2}+ct+c', \ x=bt+b'. \tag{69}$$

Taking the origin of the co-ordinates at the point A, when $t=0$, $x=0$, $y=0$, therefore $b'=0$, $c'=0$.

Let v be the velocity due to the projectile force, or the initial velocity, call the angle TAQ, as before, i. The components of the initial velocity are v cos. i, acting in the direction of X; and v sin. i, acting in the direction of y; and from (63)

$$\frac{dx}{dt}=v \cos. i, \quad \frac{dy}{dt}=v \sin. i;$$

and as the constant quantities, b and c, represent these velocities,

$$b=v \cos. i, \quad c=v \cos. i;$$

therefore by (69)

$$y=-\frac{gt^2}{2}+vt \sin. i, \quad x=vt \cos. i;$$

eliminating t,

$$y=x \tan. i-\frac{gx^2}{2v^2 \cos.^2 i};$$

but as $v=\sqrt{2gs}$ (61), $v^2=2gs$, by the substitution of which we obtain,

$$y=x \tan. i-\frac{x^2}{4s \cos.^2 i}. \tag{68}$$

A comparison of this with (67), by means of the preceding investigation, shows that this is the equation of a parabola, whose axis is parallel to the direction of the accelerating force.

The maximum value of g is VS, the length of the axis of the parabola. In this case $dy=0$, and the differential of (68) is

$$dx \tan. i-\frac{xdx}{2s \cos.^2 i}=0;$$

as tan. i. cos. i=sin. i,

$$x=2s \sin. i \cos. i;$$

and as, $2 \sin. i \cos. i = \sin. 2i$,

$$x = s \sin. 2i; \tag{70}$$

substituting this value of x in (68), we obtain

$$y = s \sin.^2 i. \tag{71}$$

The value of x (70) is equal to AS; this is half of AB, which we shall call A; therefore

$$A = 2s \sin. 2i. \tag{72}$$

CHAPTER V.

OF THE MOTION OF POINTS COMPELLED TO MOVE UPON SURFACES, UNDER THE ACTION OF ACCELERATING FORCES.

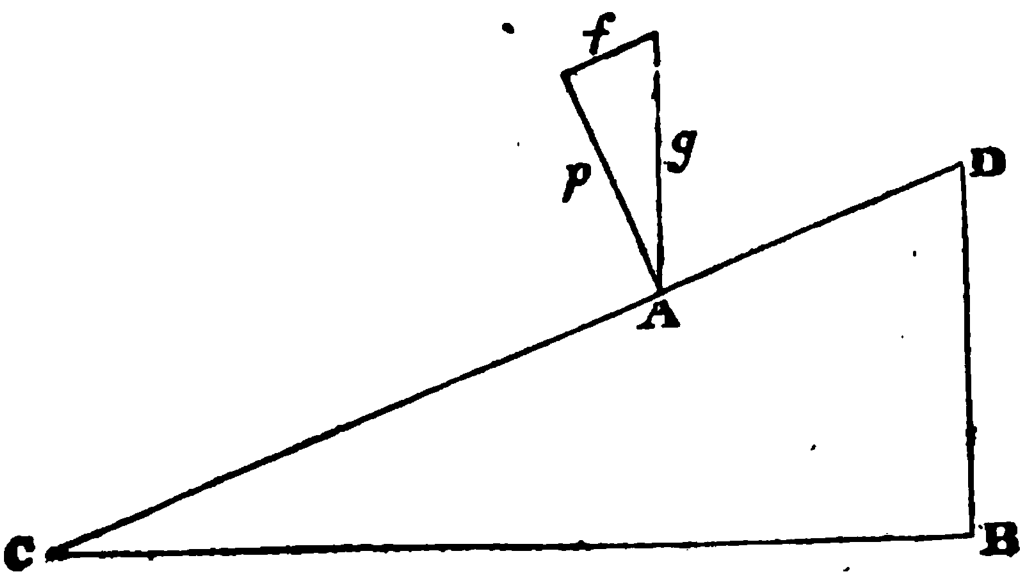
54. When a point rests upon a given surface, this surface may be considered as exerting a force to resist the pressure, which force acts in the direction of a normal to the surface. In the case of equilibrium, this is equal to the resultant of all the other forces, § 19. But in the case of motion, the action of the surface must be found, by decomposing the resultant of all the forces that act into two components, one of which is parallel, the other a normal to the surface. The last of these will represent the intensity with which the point is pressed against the surface, by the forces, and will be in equilibrio with the resistance of the surface; the former therefore, will alone remain to cause the motion of the body.

55. The simplest case that can occur is where the surface is plane, and is inclined at a given angle to the direction of an accelerating force, which is always parallel to itself. In this case, the point that is impelled by the force will describe a straight line, which is the common intersection of the surface and the plane in which the accelerating force acts. So also if a similar force act upon a point placed on a curve surface, the point will describe a plane curve, which is the common intersection of the surface and the plane in which the force acts. In these instances the surfaces may be considered as straight lines or plane curves. So also when the force is constantly directed to a point, and its directions all lie in one plane, the curve described is a plane curve. In these several instances, the circumstances of the motion may be investigated in a more elementary manner than can be done by pursuing the general method of curvilinear motion, and referring the forces to their rectangular co-ordinates. When the point moves on a plane surface, we shall continue to use the name of the surface; but in all other instances, we shall merely name the curve that is described. In our subsequent applications of the theory, the modes, in which the path of a body may be made to coincide with the several curves, will be pointed out.

56. When a point moves upon a plane surface, under the action of an accelerating force whose intensity is constant, and whose direction is always parallel to itself, that part of the accelerating force which remains to cause the motion of the point, is also a constant accelerating force, and is to the whole accelerating force

in the ratio of the cosine of the inclination of the surface to the direction in which the point, if free, would move under the action of that force.

Let A be the point, moving upon a plane surface, CD, under the action of a constant force g, whose direction makes with that of the surface the angle i'. Draw BC perpendicular to BD, which is parallel to the directions of the accelerating force, the length of the plane being CD, we shall call BD its height.



Decompose the force g into two, one of which, p, is a normal to the surface, the other, f, is parallel to it; their values will be (14)

$$p=g.\ \text{sin.}\ i',\quad f=g\ \text{cos.}\ i'.$$

The action of the former is destroyed by the action of the surface, the latter remains to move the point along it.

The motion of the point along the surface will be uniformly accelerated, for f bears a constant relation to g, which is a constant accelerating force.

All the formulæ in § 49, that have reference to uniformly accelerated motions, are applicable to this case, by substituting f, or its value g cos. i, for g; hence

$$\left.\begin{aligned} v'&=gt\ \text{cos.}\ i', \\ s'&=\frac{v'^2}{2g\ \text{cos.}\ i'}, \\ s'&=\frac{gt^2\ \text{cos.}\ i'}{2}, \\ t'&=\frac{v'}{g\ \text{cos.}\ i'}. \end{aligned}\right\} \qquad (73)$$

Comparing these with the quantities of the same kind in free motion, under the action of the same accelerating force, we have for the ratio of the velocities acquired in equal times,

$$\frac{v'}{v}=\text{cos.}\ i'; \qquad (74)$$

for the ratio of the spaces described in equal times,

$$\frac{s'}{s}=\cos.\ i'; \qquad (75)$$

for the ratio of the spaces described in attaining equal velocities,

$$\frac{s}{s'}=\cos.\ i'; \qquad (76)$$

for the ratio of the times in which equal velocities are attained,

$$\frac{t}{t'}=\cos.\ i; \qquad (77)$$

the velocity, acquired in passing freely through the height of the plane, is (61)

$$v=\sqrt{2gs},$$

in moving along the plane surface,

$$v'=\sqrt{(2g\cos.\ i'.\frac{s}{\cos.\ i'})}=\sqrt{2gs};$$

these velocities are therefore equal, or

$$v=v'. \qquad (78)$$

In plane surfaces of unequal inclinations, but equal heights, the velocities attained are therefore equal.

If the surface coincide in direction with the chord of a circle, that terminates at either end of the diameter that corresponds with the direction of the force, the time of describing the surface, and passing freely through the diameter, will be equal.

Call the diameter a,

$$t=\sqrt{\frac{2a}{g}};$$

the length of the chord is $a\cos.\ i$, and the time of describing it,

$$t'=\sqrt{\frac{2a\cos.\ i'}{g\cos.\ i'}}=\sqrt{\frac{2a}{g}}. \qquad (79)$$

And the same will be true of all chords, terminating at either extremity of the diameter which is parallel to the accelerating force.

The time of describing planes of equal inclination, is as the square roots of their lengths. Call the lengths l and l'; the times t' and t'',

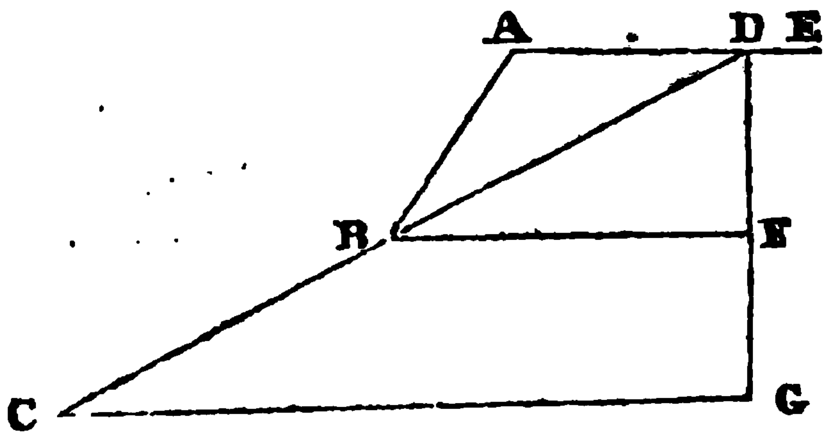
$$t'=\sqrt{\frac{2l}{g\cos.\ i}},\quad t''=\sqrt{\frac{2l'}{g\cos.\ i}},$$

$$t':t''::\sqrt{\frac{2l}{g\cos.\ i'}}:\sqrt{\frac{2l'}{g\cos.\ i'}},$$

$$t':t''::\sqrt{l}:\sqrt{l'}. \qquad (80)$$

57. If a point move from rest, upon a system of plane surfaces, under the action of a constant accelerating force, it will acquire the same velocity as it would have acquired, in moving under the action of the same force, upon a single plane surface of a height equal to that of the system; or to that it would have acquired, in moving freely through the height of the system.

Suppose first, that there are two planes, AB, BC; let AE be



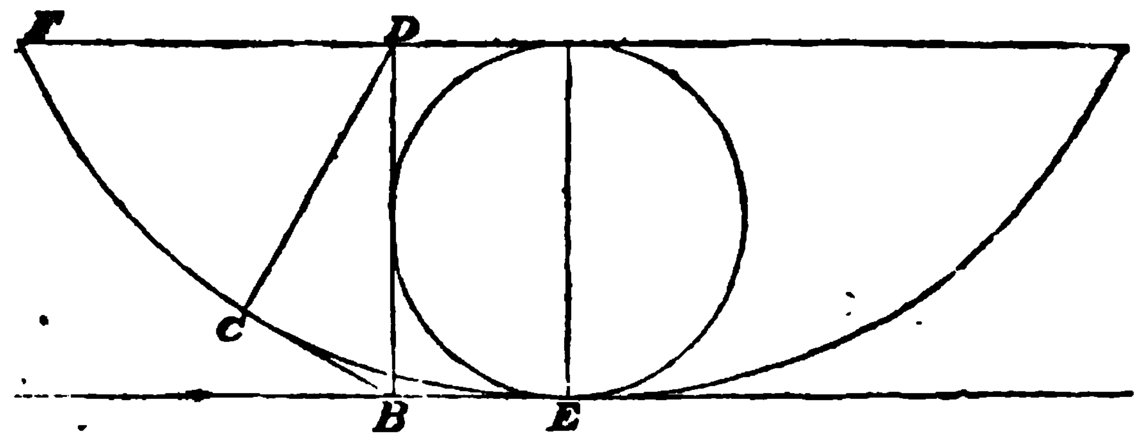
a line perpendicular to the direction of the accelerating force; produce the plane BC, until it intersect AE in D. A point moving from a state of rest under the action of the force, along AB, will acquire the same velocity as in moving freely through DF, or in moving along a surface in the direction BD; it will therefore go on in BC as if it had come from D, and will reach C with the velocity it would have acquired in describing the plane DC, or in passing freely through DG.

Had there been three planes in the system, the same mode of reasoning would have led to the same result, and so for any number, or for a curved surface.

58. In this proposition, it is obvious that any resistance that might take place at the angles of the planes, is left out of account. If the planes in the system become infinitely small, the surface becomes curved, and this proposition, therefore, holds true in respect to the arc of a curve, in which the acquired velocity is the same as that acquired in moving freely through its height.

59. A point moving in a cycloid, under the action of a constant force describes every different arc that is terminated at the extremity of the diameter of the generating circle, in equal times; and the time of describing each, is to the time of free motion, through the diameter of the generating circle, as half the circumference of a circle is to its diameter.

If the constant force g, that acts at the point C of the curve, be resolved into two components, one of which is parallel, the



other perpendicular to the tangent at that place; as the tangent will coincide with the element of the curve ds, that part which remains to cause the motion in the curve, will be to the whole accelerating force, as CB to DB. But of these, DB is a constant quantity, and BC is half of the arc CE. Hence the force that acts at any part of the curve, is proportioned to its distance from the lowest point. Now if two different arcs of the curve, both terminating at the point E, be supposed to be divided into an equal number of elements, in each of which the velocity is constant; the forces will be proportioned to these elements, and they will in consequence be described in each case in equal times; and as the number of elements are equal, the sums of these times will be equal also, and both arcs will be described in equal times. The whole semicycloid, FE, will therefore be described in the same time with any of its smallest arcs.

The time of describing the semicycloid is easily found from the general expression of variable motion, (53),

$$v=\frac{ds}{dt};$$

whence

$$dt=\frac{ds}{v}; \tag{81}$$

but for the semicycloid, the diameter of whose generating circle is a,

$$v=\sqrt{2ga}$$

$$s=\pi\frac{a}{2};$$

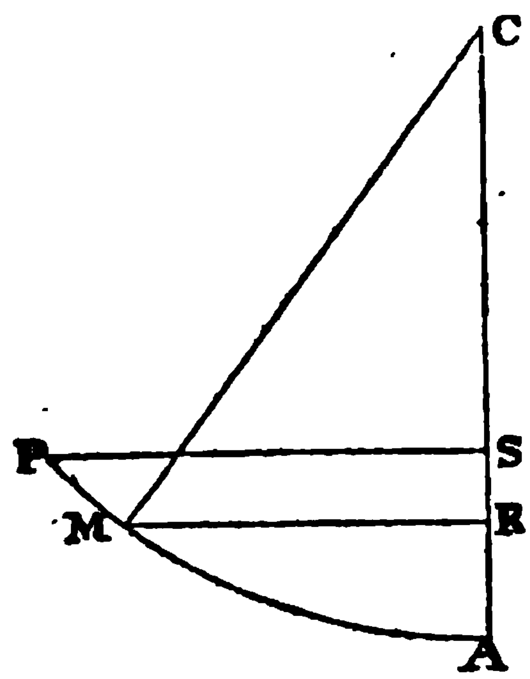
we therefore obtain for the integral of this expression (81)

$$t=\frac{\pi}{2}\sqrt{\frac{a}{g}} \tag{82}$$

π being the ratio of the circumference of a circle to its diameter. The time of the fall through the radius of the generating circle is $\sqrt{\frac{a}{g}}$; to which the last value of t has the ratio of $\frac{\pi}{2}:1$.

60. If the body describe a circular arc of very small size, the time of describing it is the same as that in which an arc of a cycloid is described, if both terminate at the point where the direction of the force is perpendicular to the curve; but if the arcs have any amplitude, the time of describing the circular arc will be greater than that of describing the arc of a cycloid, by the quantity $\frac{h}{8a}$; in which h is the versed sine of the arc described, and a the radius of the circle.

Suppose the body began to move from P, under the action of an accelerating force, acting parallel to the radius CA, and to have reached the point M.



Let $AR=x$; $RM=y$; $AM=s$; the radius $CA=a$, and $AS=h$; we have for the point M, from the nature of the circle,

$$ds=\frac{adx}{\sqrt{(2ax-x^2)}};$$

and the velocity at M being due to the height SR,

$$v=\sqrt{2g(h-x)}.$$

g being the measure of the constant accelerating force; hence

$$dt=\frac{ds}{v}=\frac{a}{\sqrt{2g}}\cdot\frac{dx}{\sqrt{(h-x).(2ax-x^2)}},$$

which can only be integrated by means of a series. For this purpose the equation is resolved into the following form,*

$$dt=\frac{\sqrt{a}}{2\sqrt{g}}\cdot\frac{dx\left(1-\frac{x}{2a}\right)^{-\frac{1}{2}}}{\sqrt{(hx-x^2)}};$$

which developed, and integrated from $x=h$ to $x=0$, gives,

* See Venturoli, Poisson, and Laplace.

$$t=\frac{\pi}{2}\sqrt{\frac{a}{g}}\left\{1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1.3}{2.4}\right)^2\frac{h^2}{4a^2}+\text{\&c.}\right\}. \quad (83)$$

In small arcs the first term of the series is alone necessary, and the expression becomes

$$t=\frac{\pi}{2}\sqrt{\frac{a}{g}}\times(1+\frac{h}{8a}); \quad (84)$$

in very small arcs the series vanishes altogether, and

$$t=\frac{\pi}{2}\sqrt{\frac{a}{g}}. \quad (84\ a)$$

61. When the accelerating force, instead of acting parallel to itself, is directed to a fixed point, it is called a central force.

It will be obvious from what has already been said, that a point acted upon by a projectile force, and afterwards drawn towards a centre by an accelerating force, must describe a curve.

The curve thus described is called a Trajectory, or Orbit.

If the point after moving completely around the centre of force, again describe the same path, the orbit is said to be re-entering.

The time in which a re-entering orbit is described, counting from the instant in which the moving point sets out from a given position in the curve, until it return to it again, is called the Periodic Time.

A line drawn from any point in the orbit is called a Radius Vector.

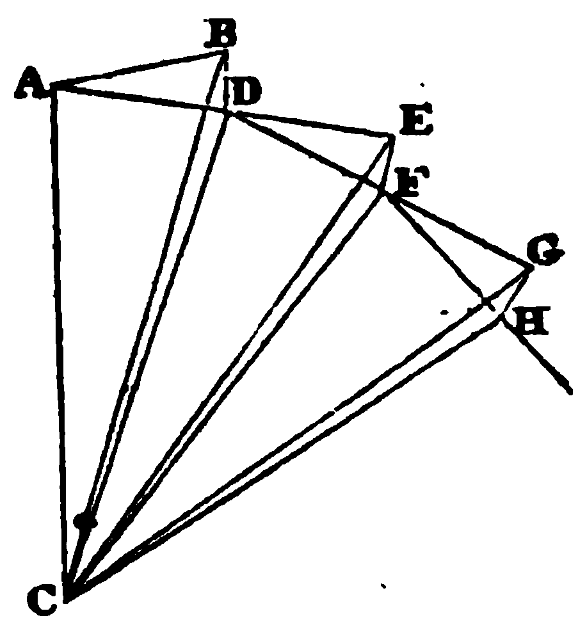
62. The simplest case of central force is, where a body, connected to a fixed point by an inflexible straight line, is impelled by a projectile force at right angles to that line. The latter force would have impressed upon the body, a motion with an uniform velocity. The body will then, in consequence of its connexion with the fixed point, describe a circle, of which that point is the centre. If the connexion were to cease at any point in the curve, the deflecting force would cease to act, and the body would go in a straight line, whose direction would be a tangent to the curve. The force acting at any point in the curve, must therefore be decomposed into two, one of which is in the direction of the curve, the other in that of its radius. The last of these is called the centrifugal force, and is equal and directly opposite to the force that draws the body towards the centre, and which, for distinction sake, is called the centripetal force.

In elementary treatises, the consideration of the action of central forces, is usually limited to the case of motions in circular orbits, described under circumstances analogous to those we have recited. If instead of an inextensible straight line, an attractive force equal and directly opposed to the centrifugal force that has

just been defined, be substituted, the circumstances will be the same as if the connexion were made by the inextensible line, and the body will in like manner describe a circle. The relations among the forces, velocities, times, and spaces, that are found in respect to circles, may be applied to the case of other curves; for a motion in a curve may, for a short space of time, be considered as corresponding with that in a circle, whose radius is the radius of curvature of that part of the curve.

63. In describing a circular orbit, under the action of two forces, one of which is projectile, the other constant, and directed to the centre of the circle, a point will move with uniform velocity.

Let the point be situated in the curve at A, and suppose a pro-



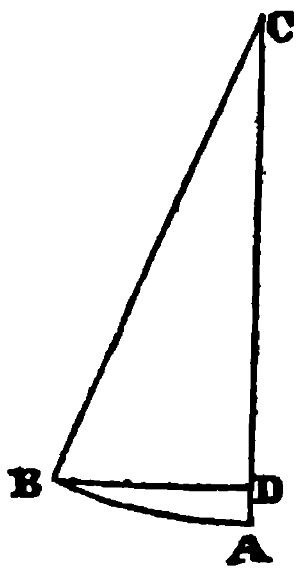
jectile force to act, which in an infinitely small time, would carry it forward in the direction AB, to the point B; the central force would in the same time cause a deflection to D. If the central force were not to cease to act, the point would go on in the direction AD produced, and describe in an equal portion of time a space DE equal to AD; but the central force continuing to act, brings it, at the end of an equal interval, to the point F; and in this way it would be found at the end of the third interval at H. Now it is obvious that the several triangular spaces CAB, CAD, CDE, CDF, CEG, CFH, are all equal, and hence the radius vector describes equal areas in equal times; these areas are however in fact, sectors of the curve; and in a circle, whose radii are all equal, if the centre be also the centre of force, the arcs which are described by the moving point, are also equal in equal times; in unequal times, proportioned to the times; and the circle is described with uniform velocity.

64. In circular orbits, the force is represented by the square of the velocity divided by the radius.

The projectile velocity is such as would have been acquired by falling freely, under the action of the central force, through half the radius of the circle.

If points revolve in different circles, the central forces are directly as the radii of the orbits, and inversely, as the squares of the periodic times; if the times be equal, they are therefore as the radii; if the radii be equal, they are inversely as the squares of the periodic times.

Let a point describe a very small arc, AB, in the element of the time dt. During this small period, the versed sine, AD, will be the space described under the action of the central force; this force may therefore be measured by the velocity that would have been acquired in moving through AD with accelerated motion; the formula (61) gives



$$v=\frac{2s}{t^2},\ \text{whence}\ f=\frac{2s}{t^2}.$$

In this expression, $2s$ is twice the versed sine of AB; the very small arc, AB, may be considered as coinciding with its chord, and the chord is a mean proportional between the radius and the versed sine; call the arc AB, a, and we have

$$2s=\frac{2a^2}{2r}=\frac{a^2}{r},$$

and

$$f=\frac{a^2}{r}:t^2=\frac{\left(\frac{a}{t}\right)^2}{r};$$

now as the velocity in the circle is uniform,

$$v=\frac{a}{t}\ \text{and}\ v^2=\left(\frac{a}{t}\right)^2; \tag{85a}$$

whence

$$f=\frac{v^2}{r}. \tag{85}$$

To compare the intensity of this force with that of the central force g, supposed to be constant; from (61)

$$v^2=gs,$$

whence

$$f=\frac{2gs}{r},\quad \frac{f}{g}=\frac{2s}{r}; \tag{86}$$

and when f is equal to g,

$$2s=r;$$

the space then to which the velocity is due, is equal to half the radius.

Let T be the periodic time, the circumference of the circle is πr, and (85*a*)

$$v=\frac{2\pi r}{T}; \tag{87}$$

substituting this value of v in that of f, we have

$$f=\frac{4\pi^2 r}{T^2}; \tag{88}$$

whence the forces are directly as the radii of the orbits, and inversely, as the squares of the periodic times.

65. If the central force be such as varies in the inverse ratio of the squares of the distances, the squares of the periodic times are proportioned to the cubes of the distances.

If two forces, F and f, act in different circles, with intensities inversely proportioned to the squares of the distances, by (85)

$$F=\frac{V^2}{R}, \quad f=\frac{v^2}{r}, \text{ and}$$

$$F:f::V^2:v^2;$$

but by hypothesis,

$$F:f::r^2:R^2;$$

therefore

$$\frac{V^2}{R}:\frac{v^2}{r}::r^2:R^2;$$

whence

$$V^2:v^2::r^3:R^3;$$

and as the times are inversely as the velocities,

$$T^2:t^2::R^3:r^3, \tag{89}$$

or the squares of the periodic times, are as the cubes of the radii.

66. Having thus investigated the circumstances of motion, in those lines and curves that will be of most frequent application in practice, by more elementary processes; we shall next exhibit the general principles that are applicable to any case whatsoever, by means of the resolution of all the forces into three, parallel to three rectangular co-ordinates.

Call the resistance of the surface N; the angles that the rectangular axes make with the normal to the surface a, b, c; the components of the resistance will be (18)

$$N\cos. a, \quad N\cos. b, \quad N\cos. c; \tag{90}$$

the values of the three rectangular components of the active forces are by (62)

$$X=\frac{d^2x}{dt^2}; \quad Y=\frac{d^2y}{dt^2}, \quad Z=\frac{d^2z}{dt^2};$$

to each of which must be added the component of N that acts in its direction; whence we have, for the equations of motion upon a curve,

$$\left.\begin{aligned} X+N\cos. a&=\frac{d^2x}{dt^2}; \\ Y+N\cos. b&=\frac{d^2y}{dt^2}; \\ Z+N\cos. c&=\frac{d^2z}{dt^2}; \end{aligned}\right\} \qquad (91)$$

the values of the velocities from (63), are

$$\frac{dx}{dt}, \quad \frac{dy}{dt}, \quad \frac{dz}{dt};$$

these may be reduced to the direction of the normal, by multiplying each by the cosine of the angle it makes with that line; and as the direction of the motion is at right angles with the normal, the sum of these three components must be 0, or

$$\frac{dx}{dt}\cos. a+\frac{dy}{dt}\cos. b+\frac{dz}{dt}\cos. c=0. \qquad (92)$$

We may eliminate the unknown quantities in the three equations, (91) by adding them together, after multiplying the first by dx, the second by dy, and the third by dz; the sum of this addition becomes, if we take into account the equation (92),

$$\frac{dxd^2x+dyd^2y+dzd^2z}{dt^2}=Xdz+Ydy+Zdz. \qquad (93)$$

If the quantity $Xdx+Xdy+Xdz$, is the exact differential of the three variable quantities, x, y, z, we obtain by integrating

$$\frac{dx^2+dy^2+dz^2}{dt^2}=2f(x\,,y\,,z)+C; \qquad (95)$$

now as $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are the components of the velocity in the direction of the three axes; the first member of this expression represents the square of the velocity, and

$$v^2=2f(x\,,y\,,z). \qquad (96)$$

It may be remarked that there are several cases in practice, where the expression $Xdx+Ydy+Zdz$, is not an exact differential of three variable quantities; such for instance is the case, where a body is acted upon by a fluid resistance, or by friction.

Where the three rectangular forces, X, Y, and Z, are each equal to 0,

$$f(x\,,y\,,z)=0,$$

and

$$v^2=C,$$

or the square of the velocity, and consequently the velocity itself,

is constant. Thus, then, when the accelerating force ceases to act, the body will continue to move with uniform velocity.

If the forces, X, Y, and Z, retain a determinate magnitude, the velocity is no longer constant, but it is independent of the curve the body is compelled to describe; for if we call A the velocity that corresponds to another point of the curve, whose co-ordinates are, a, b, c; we can infer the same in respect to this, as to the original case, and

$$A^2=C+2f(a\,,\,b\,,\,c),$$

which substracted from the former equation, gives

$$v^2-A^2=2f(x\,,\,y\,,\,z\,,)-2f(a\,,\,b\,,\,c)\,; \qquad (97)$$

it is obvious therefore, that the change in the squares of the velocities, has no relation to the form of the curve described between the two points. So also, when several bodies set off from a given point, under the action of the same accelerating force, they will all, on reaching another point, have the same velocities, however various may be the lines described in the mean time, and different the times of describing them. This is an obvious generalization of what was found, in § 58, to happen, when the motion took place in plane curves, under the action of a constant force.

The further pursuit of this subject, by higher methods of analysis, leads to a remarkable law which is called The Principle of the Least Action. It may be thus expressed:

When a body is acted upon by forces whose relation is expressed by the formula, (95) and the curve is not determined, it will choose for the direction of its motion the shortest line that can be drawn on the surface, or that in which the integral, $\int vds$, is a minimum.

When the accelerating force ceases to act, the velocity is, as we have seen, constant, and

$$\int vds=vs,$$

and as s is the shortest space, the body will pass from one point to another, in the shortest possible time.

For the further illustration of this subject, we refer to Bowditch's translation of Laplace.

67. When a point is acted upon by but one accelerating force, constantly directed towards a fixed point, the path will be a plane curve; and the areas, described around this point by the radius vector, are proportioned to the times employed in describing them.

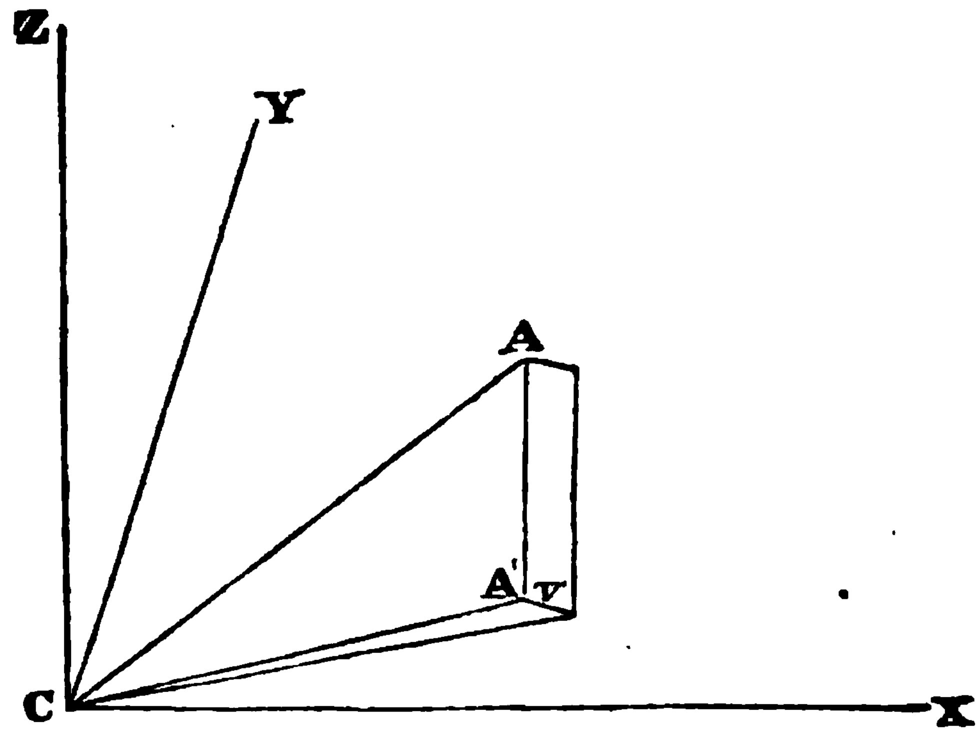
By (62),

$$\frac{d^2x}{dt^2}=X\,,\ \frac{d^2y}{dt^2}=Y\,,\ \frac{d^2z}{dt^2}=Z\,;$$

these are readily transformed into the following

$$xd^2y-yd^2x=(xY-yX)\,.\,dt^2$$
$$zd^2x-xd^2z=(zX-xZ)\,.\,dt^2$$
$$yd^2z-zd^2y=(yZ-zY)\,.\,dt^2.$$

Let us suppose the origin of the co-ordinates to be at the centre



of force C, and let the line, CA, which joins the point in which the body is at any given time, to the centre, represent the central force. It will be obvious that the three sides of the parallelopiped which represent the three rectangular components of CA, will be the co-ordinates of the point A, and for any other magnitude of the central force, they will be proportioned to these components; hence,

$$X : Y : Z :: x : y : z,$$

and

$$xY=yX\ ,\ z\ X=xZ\ ,\ y\ Z=zY$$
$$xY-yX=0\ ,\ zX-xZ=0\ ,\ yZ-zY=0,$$

and in the former equations the second members become $=0$; integrating these, we have

$$xdy-ydx=cdt$$
$$zdx-xdx=c'dt$$
$$ydz-zdy=c''dt.$$

If these expressions be multiplied, the first by z, the second by y, the third by x, they may be added, and we have

$$cz+c'y+c''z=0,$$

which is the equation of a plane surface; whence we may infer that the orbit is a plane curve. Take now the projection of the point A, on the plane of x, y; let A' be this projection; the radius vector, C $A'=r$; and v, the angle that determines its direction, in respect to the axis parallel to x; by (18)

$$x=r\ \cos.\ v,\ y=r\ \sin.\ v,$$

whence

$$xdy-ydx=r^2dv\ ;$$

and as the first number of this equation is, as we have seen, a constant quantity, so also will be the second. But we may suppose

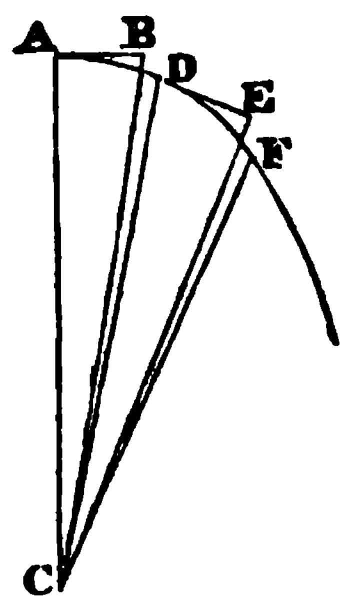
the arc of the curve, described by the body in passing between the two successive positions, marked out by the small angle dv, to be a circle; and the area described by the radius vector as a circular sector whose area is $\frac{r^2dv}{2}$, and this being the half of a constant quantity, is itself a constant quantity; it will also be equal to $\frac{cdt}{2}$, and consequently, the area described in t is $\frac{ct}{2}$, wherefore the area described during a given time, is proportioned to the times. The areas then, that are described by the radius vector of the projection of a moving point, upon a plane passing through the centre of force, are proportioned to the times. Now as the orbit of the body is itself a plane curve, and as the direction of the plane of projection is arbitrary, this proposition is equally true in respect to the areas described by the radius vector of the body itself, upon the surface of the plane curve which forms the orbit.

This proposition is the same which has been investigated in § 63. And is not only true of circular orbits, but of any orbits whatsoever.

68. This same proposition may be deduced from more simple considerations. In the first place, the curve described under the action of a central force, must be a plane; for, the direction of the motion, in the first element of the time, will be in the plane formed by its two components, the projectile, and the central force; the direction in which the moving point would tend to go on, were the central force to cease to act, will likewise be in the same plane, and the line, which joins the extremity of this tangential force to the centre, will be in this plane also; this line represents the direction of the central force, during the second element of the time, and the resultant is therefore still found in the original plane; and so of all the successive resultants, which make up the curve, which lies therefore in one plane.

We have seen from § 33, that the sum of the Moments of Rotation of any number of forces, is equal to the Moment of Rotation of their resultant. Now if we suppose the curve to be a polygon of an infinite number of sides, and the velocities in each of these to be uniform, the spaces described will be the measure of the several forces; the moment of rotation of the projectile force will be the tangential line AB, which represents that force,

multiplied by the radius vector CA, or twice the triangle ABC; the moment of rotation of the force that acts in the curve, is the rectangle under AC, and AD, or twice the sectoral space, ADC. And as the direction of the central force is in the radius vector, its moment of rotation $=0$; hence the two moments of rotation, of the tangential force, and of the force in the curve, are equal, as are also their respective halves. At the point D, the tangential force is that with which the curve has been described; its moment of rotation is therefore equal to twice the space DCE, and for the same reason, it is equal to twice the second sectoral space, ADC, which is therefore equal to the sectoral space, ADF; and these are the spaces described by the radius vector, in the equal elements of the time; hence the greater spaces which these make up, will be proportioned to the times.



CHAPTER VI.

Principle of D'Alembert.

69. The formulæ and conditions of the motion and equilibrium of points, may be applied to systems of material points, or to such bodies as actually exist in nature, by means of a self-evident principle, first announced by D'Alembert, and which has been employed by all succeeding writers on Mechanics of any well founded reputation. It may be expressed as follows:

If there be a body, or system of points materially connected in any manner with each other, and which are acted upon by forces given in magnitude and direction; the action of these several forces is modified by the connexion among the several points, and they neither move in the direction, nor with the velocity they would have, were they not connected. Still the forces that must be compounded with those that cause the motion, in order to make up the forces with which the points actually move, must be such as are in equilibrio with each other, or that if they acted upon the system alone, would produce no motion. The last mentioned forces obviously represent the mutual action of the points upon each other; these could not of themselves cause motion, and are therefore in equilibrio.

To express this analytically, let the forces applied to the points be represented by the velocities a, b, c, &c.; the velocities the points actually have, by a' b', c'; the velocities which must be combined with the first of these, in order to produce the last, a'', b'', c'', &c., then

$$a''+b''+c''+\&c.=0;$$

for it is obvious that these velocities have no effect upon the actual moving force of the system, which is due to the velocities a, b, c, &c., alone; and hence the forces which they represent mutually destroy each other.

To apply this to systems of bodies, we must consider that the moving force of each will depend not only on its velocity, but on the number of equal material points it contains; therefore, the force, by which each body is impressed, is due to the product of its number of points into its velocity; calling the former, A, B, C, &c., we have

$$\mathrm{A}a''+\mathrm{B}b''+\mathrm{C}c''+\&c.=0. \qquad (98)$$

CHAPTER VII.

Principle of Virtual Velocities.

70. If we suppose the equilibrium of any system of forces whatsoever to be disturbed for an instant, and that each point to which a force is applied, has a velocity, such as would have been given to it, in the direction of the disturbance, by the force which acts upon it. The velocities that the several points would acquire, are called their Virtual Velocities.

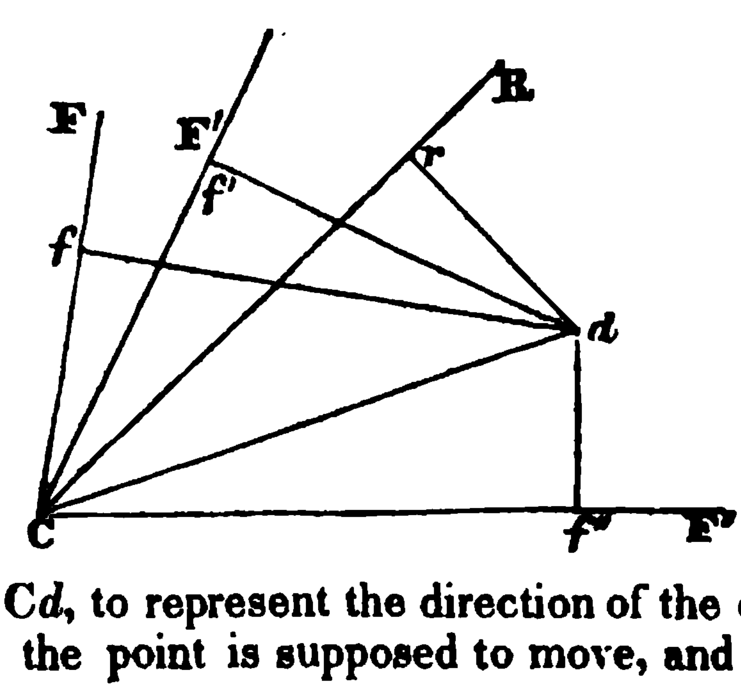
71. In any system of forces whatsoever, the sum of the products of the several forces, into their respective virtual velocities, is equal to 0, or calling the forces, F, F′, F″, the virtual velocities, v, v', v'',

$$Fv+F'v'+F''v''+\&c.=0. \qquad (99)$$

Forces can only act, under certain definite circumstances, upon their points of application.

1. All the forces may be applied to a single free point.
2. The point of application of the forces may be compelled to rest upon a given surface.
3. It may be compelled, when it does move, to move along a given surface.
4. The forces may act upon a system of points, united together in any possible manner.
5. The system may have a fixed point, around which the others must move, or a point that is compelled to describe a given surface.

First. Let the forces act upon a single point, C, in the figure beneath; let FC, FC′, FC″, &c., be their several directions and RC, the direction of their resultant R.



Draw a line, Cd, to represent the direction of the disturbance, or that in which the point is supposed to move, and the distance to

which it is removed. This line will therefore represent the virtual velocity of this point. Decompose each of the forces into two others, one parallel to the line Cd, the other perpendicular to it; the respective values of the components will be, calling the angles that they respectively make with Cd, a' and b, b', b'', &c.,

$$\text{R} \cos. a, \ \text{F} \cos. b, \ \text{F} \cos. b', \ \text{F} \cos. b'', \ \&\text{c}.$$

Now if perpendiculars be let fall from the point d, upon the several directions of these forces, the angles these will respectively make with the line Cd, will be equal to the angles the directions of the forces make with it at the point C. The value of the projections, r, f, f', f'', &c., will therefore be symmetric with the values of the components of the forces parallel to Cd, or calling the line Cd, c,

$$r=c \cos. a, \ f=c \cos. b, \ f'=c \cos. b', \ \&\text{c}.$$

Now, as the sum of the components of any force, estimated in a given direction, is by (19) equal to the component of the force estimated in the same direction,

$$\text{R} \cos. a=\text{F} \cos. b+\text{F}' \cos. b'+\text{F}'' \cos. b''+\&\text{c}.$$

multiplying this by the value of c, obtained from the foregoing expressions, we obtain

$$\text{R}r=\text{F}f+\text{F}'f'+\text{F}''f''+\&\text{c}.$$

but r is the virtual velocity of the point C, and f, f', f'', &c., are obviously the components of that velocity, in the several directions of the component forces, and are therefore so much of the virtual velocity as is due to the action of the respective forces; substituting then the letters v, v', v'', &c. for f, f', f'', &c., and supposing the system to be in equilibrio, we obtain

$$\text{F}v+\text{F}'v'+\text{F}''v''+\&\text{c}.=0.$$

Second. In the case of the system of forces being such as causes a point to rest upon a given surface, the action of this surface may, § 19, be represented, by introducing a force normal to the surface in its direction, and equal to the resultant of the other forces, which is also a normal to the surface. This force may therefore be introduced among the forces in the above equation; calling it P, and the virtual velocity it causes, p, we have

$$\text{P}p+\text{F}f+\text{F}'f'+\text{F}''f''+\&\text{c}.=0;$$

and

$$\text{F}f+\text{F}'f'+\text{F}''f''+\&\text{c}.=0;$$

the proposition is therefore true in respect to points compelled to rest upon a surface by the action of the forces.

Third. The same reasoning applies to the case where the point is compelled, if it move, to move along a given curve; the motion being very small, may be considered as taking place in the direction of the tangent to the curve, or perpendicular to the di-

rection of P; the virtual velocity of p is therefore equal to 0, for the angle of inclination becomes 90°; and

$$Pp=0;$$

hence again,

$$Ff+F'f'+F''f''+\&c.=0.$$

Fourth. If the forces act upon points composing a system, in which they are united in any manner whatsoever; call the forces, which we shall for the present restrict to three in number, F, F', F''. Each point is held in its position, in the case of equilibrium, by actions exerted in consequence of its connexion with the other points in the system; these actions may be represented by forces, coinciding, in their several directions, with the lines that join each point to the others which compose the system, and these forces will have a determinate magnitude. Call the forces, that represent these mutual actions, on the three several points, A, A', B, B', C, C'. Now as each point will be in equilibrio under the action of three forces, that which is applied to it, and the two which are exerted upon it by the others, the principle of virtual velocities holds good in respect to each of these three sets of forces, or

$$Ff+Aa+A'a'=0,$$
$$F'f'+Bb+B'b'=0,$$
$$F''f''+Cc+C'c'=0;$$

but the points being mutually connected,

$$A'=B,\ B'=C,\ C'=A;$$

hence

$$Ff+Aa+Ba'=0,$$
$$F'f'+Bb+Cb'=0,$$
$$F''f''+Cc+Ac'=0;$$

adding these equations,

$$Ff+F'f'+F''f''+A(a+c')+B(a'+b')+C(a'+c')=0.$$

But the equilibrium of the system would subsist, if the forces F, F', F'', were suppressed, and the system supported by forces equal and opposite to A, A', B, B', C, C', applied to the several points; for each of these pairs of forces has for its resultant a force equal and opposite to F, F', and F'' respectively. The principle of virtual velocities is therefore applicable to this new set of forces also; or

$$-A(a+c')-B(a'+b)-C(a'+c)=0;$$

subtracting this equation from the former, we have

$$Ff+F'f'+F''f''=0.$$

And although our reasoning has, in order to prevent complexity, been restricted to three points and three forces; it is obvious that it might have been extended to any number whatsoever.

As we have assumed the points to be connected in any manner whatsoever, it is obvious that the principle is applicable to all cases of free systems of material points, whether they be united by invariable and inflexible lines, or by the loose aggregation that takes place in the particles of fluids, or by any intermediate connexion between that which is fixed and invariable, and that in which the connexion is about to cease altogether.

Fifth. In the same manner, precisely, in which the case of forces, acting upon a single free point, has been applied to those of forces acting upon a point compelled to rest or to move upon a given surface, may the above case be extended to those of systems that have in them a fixed point, or a point compelled to move upon a given surface, and thus the principle of Virtual Velocities may be shown to be true in all possible cases.

BOOK III.

OF THE EQUILIBRIUM OF SOLID BODIES.

CHAPTER I.

GENERAL PROPERTIES OF MATTER. DIVISION OF NATURAL BODIES. MEASURE OF THE MOVING FORCES OF BODIES.

72. In order to apply the preceding principles to the circumstances which occur in nature, it becomes necessary, that we should investigate not only the general abstract properties of matter, but also those which we find inherent in the greater part, if not the whole, of these material substances that we can make the objects either of experiment or observation.

These essential properties of matter, without which we cannot conceive it to exist, are, Extension, Mobility, and Impenetrability.

73. Matter must occupy a certain portion of space, and thus be extended in three dimensions, or is of that class of Geometric magnitude, which is called a Solid.

74. Being extended, it is of course capable of division, and were we to reason in respect to this secondary property, as if the material substance had the same geometric properties with the space it occupies, it might be demonstrated that matter is infinitely divisible; and thus, in many works on Mechanical Philosophy, it has been inferred, from strict mathematical reasoning, that this is the fact. It is however, obvious, that the reasoning which is applicable to mere geometric magnitude, has reference only to the space occupied by matter, and not to matter itself. We may, therefore, reasonably doubt whether matter be infinitely divisible.

Still, the divisibility of matter may be carried to an extent, that exceeds the limits that our senses can reach: thus by mere mechanical division, as in the hammering and drawing of metals, the divisibility of the substances is very remarkable; gold, when manufactured into leaf, is beaten out, until the thickness of the sheet does not exceed $\frac{1}{282000}$th part of an inch.

In the mixed mechanical and chemical arts, divisibility is carried still farther; in making gold lace, for instance, where a rod of silver is gilt by means of an amalgam, and then drawn into wire, a single grain of gold covers 9600 square inches; and when the wire is flattened, the surface covered is nearly doubled in extent.

Chemistry furnishes still more marked instances of divisibility; particularly in the manner in which the colours produced by chemical tests and re-agents, are diffused throughout spaces of considerable magnitude, by quantities of substances that are hardly appreciable:

There are also substances, that we know by our senses to exist, or can distinguish by the effects they are capable of producing, which escape the nicest methods of chemical research; thus, the matter of which many of the most powerful odours is composed, has not been detected by any analysis of the air which is its vehicle; and the pestilential miasmata which cause disease, and even certain death, have baffled the most powerful, as well as the most delicate methods of inquiry:

Another proof of the great divisibility of matter, may be drawn from the minuteness of the living animals that can be distinguished by the aid of the microscope; Leuwenhoeck saw animals not exceeding the ten thousandth part of an inch in length, and each of these is furnished with organs, has vessels in which fluids circulate, whose particles therefore must be small beyond all conception.

Mere physical action is also capable of exhibiting a very extensive divisibility, in the matter that is subject to it; thus water, when heated beyond a certain temperature is converted into steam, which occupies about seventeen hundred times as much space as the water whence it is generated, and yet no perceptible portion of the space is devoid of aqueous matter.

75. The term Body, is employed to denote a determinate quantity of matter, contained under some known figure, or existing in some peculiar mode. By mechanical, physical, or chemical action, influencing the divisibility of matter, a body may be disintegrated; its component parts will assume new forms and characters, and constitute new bodies. By such forces constantly acting in nature, the surface of the globe, and all the bodies that we find upon it, are undergoing continual changes; and are constantly assuming new forms, or entering into new combinations. In all these changes, however, no portion of matter is annihilated, but its quantity continues invariable. We know in fact of no agent in nature, that is capable of increasing or diminishing the amount of the matter that exists in the Universe. Matter, therefore, might be inferred to be eternal; but as we find that it is

not only a metaphysical truth, but one that the process of induction enables us to deduce, from a full investigation of all the phenomena of nature: that, nothing can exist in any state whatever, without some sufficient reason; we have a right to infer, that the permanence of matter, is the result of the action of a great and all powerful first cause.

76. The indestructibility of matter might be urged as a proof that it cannot be infinitely divisible, for that property could only result in its annihilation. But we are enabled to draw more conclusive evidence of the fact, that the divisibility of matter is finite, from the modern discoveries of chemistry. In this science, we are enabled to discover, and explain many important facts, by means of the hypothesis, called the Atomic Theory; this holds, that all bodies are finally resolvable into particles, incapable of farther division. From what has been said, it will however appear, that these atoms must be so small as to escape the most acute of our senses, even when furnished with the most powerful aids that the high improvement of the arts at present affords. The proof of the existence of atoms, therefore, is not, and probably cannot be, complete; for to constitute a theory that can be received as absolutely true in philosophy, it is not only necessary that it should fully explain all the phenomena, but that it should be founded on evidence entirely independent of them. Still, as in the most minute state of division, to which bodies can be reduced in practice, we find them retaining their peculiar and individual properties, we have a right to infer, that their divisibility is in no case carried to an infinite extent; we may, therefore, assume, that they are made up of portions infinitely small; these are usually called Particles.

77. The term Impenetrability, as applied to denote a property of matter, merely implies, that no two particles can occupy the same space, at the same time. In this point of view, the hardest and the softest substances are equally impenetrable. Impenetrability is absolutely essential to the existence of matter, and has an experimental proof in its indestructibility; for, were it not impenetrable, two portions of matter might be reduced to a single one, and one of them annihilated, which we have seen to be impossible. It is to this property, of excluding all other portions of matter from the space that themselves occupy, that we have recourse, in ascertaining the material existence of certain substances, concerning which, inaccurate notions long prevailed.

78. The property of mobility is also deduced from experiment and observation, for we find no body that cannot be set in motion, by an adequate force, nor any whose motion cannot in like

manner, be arrested. It might also be shown to be a consequence of the Impenetrability of matter, for as a body resists the entrance of any other, into the space itself occupies, motion must arise whenever a body, impelled by any force whatever, tends to enter the space another previously occupies.

The sum of all the particles in a body, constitutes its mass, or makes up its quantity of matter ; and in homogeneous bodies, the quantity of matter is obviously proportioned to their respective bulks.

In heterogeneous bodies, the quantity of matter has respect not only to the bulk, but to the Density; the latter is the relation that the quantity of matter, in a given body, bears to its bulk.

This definition may be thus expressed:

$$D=\frac{M}{B}; \qquad (100)$$

whence we have

$$M=DB, \text{ and } B=\frac{M}{D}; \qquad (101)$$

The Densities then, are directly as the quantities of matter, and inversely as the Bulks:

The Quantities of matter, are in the compound ratio of the Densities and Bulks; and

The Bulks, are directly as the quantities of matter, and inversely as the Densities.

79. Bodies differ from each other, in the greater or less difficulty with which their particles may be separated. When the separation of the particles requires the application of a determinate force, the body is said to be solid ; when they may be divided, by a force so small as to be inappreciable, they are said to be fluid. Of fluids, some have so small a capability of having their bulk changed by pressure, as to have been considered as absolutely incompressible; such bodies are called Liquids; of them, water at ordinary temperatures is a familiar instance. Others again, are capable of being compressed, and of occupying larger spaces, when the compressing force is removed; these are styled Gases, or Elastic Fluids; of these atmospheric air, may be cited as the type.

80. Many bodies are familiarly known to be capable, under different physical circumstances, of existing in all the three different states: thus, water, when cooled below a certain temperature, passes into the solid form, and becomes Ice; while if heated, beyond a certain limit, it assumes the gaseous state, and is called Steam. As a general rule, deduced from various chemical and purely physical facts, every body in nature is capable of assu-

ming, under proper circumstances, all of these three mechanical states.

81. Heat is the great natural agent that is concerned in these mechanical changes, and it is a general rule, that heat, existing in that modification in which it is said to be latent, determines the mechanical state that bodies assume.

When the latent heat is withdrawn, the body returns to its original state; thus steam, on parting with its latent heat, becomes water; and water, on parting with its latent heat, becomes ice: hence, we infer the action of another force in opposition to that of heat; to this force, whose cause is, like that of heat, unknown to us, we give the name of the Attraction of Aggregation. These two great natural antagonist forces then, determine the mechanical states in which bodies exist.

When the attraction of aggregation predominates, the body is a Solid.

When they are equally balanced, the body is a perfect Liquid; and when the force of heat, exceeds that of the attraction of aggregation, the body becomes an Elastic Fluid.

It will be hereafter seen, that no body has a constitution that will entitle it to the name of a perfect liquid. In all known bodies of this class, the force of attraction still preponderates over that of heat, as will be made manifest when we examine the forms in which small masses of liquids arrange themselves.

82. The attraction of aggregation is only known to us as acting at insensible distances; it is therefore unnecessary to enter into any investigation of the ratio, in which its intensity diminishes, as the distance increases. Its absolute measure differs in every different body, and must hence be determined experimentally; it constitutes the strength of the materials that are employed in practical mechanics, and will, under this name, become the object of future consideration. Although the general principles of the two first books, apply equally to all bodies, whatever be their state of aggregation; still, they are modified in their action by the peculiar mechanical properties of the different classes into which we have found them to be divided. It is hence necessary to consider the Mechanics of Solid, and of Fluid Bodies, separately.

83. In our previous inquiries, forces have been considered in the abstract, and as acting upon material points of indeterminate magnitude, although we have occasionally been compelled to use the term Body. It now becomes necessary that we should be able to estimate the force by which a body or aggregate of matter is actuated. This, which is called the Quantity of Motion, may

be ascertained from the following considerations. We may obviously consider a moving body as made up of its particles, or of a number of material points, and supposing its motion to be rectilineal, every particle will be actuated by a force, equal and parallel to those which actuate the rest; the whole force then will be equal to the mass of the body, multiplied by the forces that actuate its particles. If the motion be uniform, the velocity of each of these particles is constant, and will represent the force by which it is produced; hence, the moving force of a body will be represented by its mass, multiplied by its velocity. The same will be obvious from other considerations; for it is clear, that when several bodies have equal velocities, that which has twice the quantity of matter that another has, must have twice the quantity of motion, and so on. In like manner, if the masses be equal, the body which has the greatest velocity will have a quantity of motion exactly proportioned to it.

This may be illustrated thus:

Let M be the mass of a body made up of the several particles p, p', p'', &c., and v the common velocity, the sum of the several forces by which they are actuated will be

$$pv+p'v+p''v+\&c.,$$

$$\text{or } (p+p'+p''+\&c.)\times v;$$

and as the sum of the particles is equal to the mass, the expression for the quantity of motion, f, will become

$$f=\mathrm{M}v; \tag{102}$$

whence

$$v=\frac{f}{\mathrm{M}}. \tag{103}$$

From these equations the following consequences immediately follow:

(1). The forces of moving bodies, or their quantities of motion, are represented by the products of their masses and velocities.

(2). In equal masses, the forces are proportioned to the velocities.

(3). When bodies have equal quantities of motion, the velocities are inversely as their masses.

(4). With equal velocities, the quantities of motion are proportioned to the masses.

CHAPTER II.

Attraction of Gravitation.

84. It is a fact demonstrated by universal experience, that all heavy bodies, if unsupported, fall towards the surface of the earth. This can only take place in consequence of the action of some specific force, which has been supposed to reside in the earth itself, and which has been called the Attraction of Gravitation, or more simply, Gravity.

85. The direction of this force may be ascertained by suspending a heavy body, from a fixed point, by a flexible string; the direction of the string will, as is obvious, mark the line in which a body would tend to move, under the action of the force.

Such an apparatus is called the Plumb-Line. When adapted to a ruler with parallel sides, it becomes a familiar and simple instrument, of great practical utility. The plumb-line hanging freely, being made to coincide with a line drawn along the ruler parallel to its sides, the latter also point out the direction of gravity.

If a second ruler be adapted to the lower extremity of the first, and make with it angles that are exactly 90°, the face of the second ruler will be found to adapt itself to the surface of stagnant waters; hence we infer that the direction of gravity is perpendicular to the surface of standing water. We say however, in general terms, that the direction of gravity is perpendicular to the surface of the earth.

We are, in truth, compelled to refer the direction of gravity to some more extended surface than that of the largest mass of tranquil water: for it is a question that requires examination, whether the direction of gravity, at a given place, be constant. To ascertain this, it would be necessary to have some points that we could consider absolutely fixed. Even the most stable edifice has not sufficient permanence, to afford points that can be relied upon as such. If, for instance, we were to find, that the relative direction of the plumb-line and the wall of a building had changed, we might at first sight doubt which had altered its position; for we know, that natural convulsions, or even gradual decay, may alter the direction of the firmest walls. Even the sides of a mountain could not be relied upon, for we know of mountains being affected by earthquakes.

The sea, however, apparently unstable as it at first sight appears, has, in its mean level, and in the limits within which its ebb and flow are bounded, an instance of the greatest stability that we

find on the face of our globe. Were the mean height of its surface to change, or the limits of its rise to be much extended, it would be attended with the most disastrous consequences, causing great inundations, or even another deluge. Were the sea to become still, and to be no longer agitated, either by the waves that are raised by the winds, or those which constitute its tides, it would come to rest, in a position whose surface would be a mean, between the limits within which its oscillations now take place. This surface it will hereafter be shown, must be every where perpendicular to the direction of gravity; for the present, we must receive this fact as true, from the experimental illustration that has been cited. It is this mean surface of the ocean, supposed to be produced through the body of the continents, and other portions of dry land, that we understand by the Surface of the Earth.

86. The earth being a body nearly spherical, as can be shown by a variety of astronomic facts, as well as from actual observation and measurement, all the directions of gravity converge towards its centre. They would actually meet there, were the earth a perfect sphere.

87. Ordinary observation might lead us at first sight to suppose, that bodies are very unequally influenced by the attraction of gravitation; that its action had relation not only to their masses, but was influenced also by their densities. Thus the metals and stones fall with great velocity; wood and other vegetable substances less rapidly; while feathers, down, and other similar substances, seem hardly to be affected by the earth's attraction. Others again actually rise from the surface, apparently in opposition to the direction of the attractive force, as vapour, smoke, and clouds. When, however, we consider, that some bodies which we may under ordinary circumstances see to fall, will, under others, rise upwards, we are tempted to examine whether analogous causes may not exist, to determine the floatation, or actual elevation of others. Thus, for instance, a piece of wood that falls in the open air, rises when placed in water, and floats at the surface; and even iron will do the same, when plunged in a mass of mercury. The air then, it is possible, may, by its buoyancy, prevent altogether the fall of some bodies, and even support them in the atmosphere; and may, by its resistance, retard the descent of others. Whether this be the fact or not, may be tested by means of the air pump. This apparatus, as will hereafter be seen, is capable of exhausting the greatest part of the air, from a proper vessel, called a receiver; and in the exhausted receiver of an air-pump, solid bodies of every diversity of density fall in times that are absolutely equal; while the lightest vapours, and

smoke, descend and occupy the bottom of the vessel. Thus all bodies are influenced by gravity, and the velocities of falling bodies being thus shown to be equal, when not resisted by the air, whatever be their densities, it follows, that the forces which actuate them, are proportioned to the masses, or quantities of matter.

88. As the earth is situated in free and open space, it follows from what has been said in relation to the inertia of matter, that it cannot impress motion on a body, without parting with an equal quantity of its own motion; or, in simpler terms, that it must move towards the falling body, with a quantity of motion, equal to that with which the body moves towards it. But the mass of the earth is so vast in respect to the bodies which we can perceive to fall, that the velocity of the earth towards the falling body will be wholly inappreciable. We cannot therefore test by observation, the fact, whether the fall of a heavy body is caused by the mutual action of the earth and the body, or whether the body alone moves. Neither can we admit mere reasoning to decide whether the attraction be mutual, or is exerted solely by the greater mass, which is, in this case, the earth.

89. If the attraction between a falling body and the whole mass of the earth be mutual, it will follow, that it must also be mutual between the parts that make up the mass; and the attracting force of the earth will be due to the sum of the separate attracting forces with which its particles are endowed. A force thus made up, will be influenced by irregularities on the surface of the earth; and large projecting masses, such as mountains, would cause a deflection in the plumb-line, from a perpendicular to the general surface of the earth.

Whether such a deviation do occur, can only be determined by the aid of astronomic observation. The deviation must be at most extremely small, for the size of the largest mountains bears but a very small proportion to the whole mass of the earth: it is therefore wholly imperceptible, except to the most accurate modes of observation, and it is only in mountains of considerable size, that it can be detected, even by them. But if such deviation of the plumb-line, from the true vertical shall be detected, it furnishes full and complete evidence, that the attraction of gravitation is mutual between the bodies that are affected by it.

The deflection of the plumb-line by the attraction of a mountain, was first suspected by Bouguer, one of the French academicians who were sent to Peru for the purpose of measuring an arc of the Meridian. In carrying on a series of triangles, in the neighbourhood of the great mountain of Chimborazo, latitudes were determined by means of observations of the stars, both on

the north and south sides of the mountain ; the itinerary measure between two of these stations, was not found to correspond to the difference of latitude, and as the apparent latitude depends on the position of the zenith, pointed out by the plumb-line, this discrepancy showed a deflection in the latter. The deflection observed by Bouguer, did not appear to exceed 7″ or 8″.

In 1772, Maskelyne, the British Astronomer Royal, performed a series of observations of the same character, at the base of the mountain Schehallien, in Scotland. The result of these was, to show conclusively, the fact of the deviation of the plumb-line; and from his having this sole object in view, and from the accuracy of his observations, we can place the most implicit confidence in his inferences. The deviation was found to amount to 54″.

It will be at once seen, that, if the density and bulk of the mountain be known, and the bulk of the earth, this experiment affords a ready mode of determining the mass, and consequently the density of the earth ; for the plumb-line will point out the direction of the resultant of two forces, one of which is the attractive force of the earth, the other that of the mountain ; and the angle of deviation will enable us to calculate one of these, when the other is given, upon the principles of the composition and resolution of forces in § 12.

Professor Playfair performed the geological examination that was necessary to determine the nature of the materials composing the mountain. Professor Hutton, upon these data, proceeded to calculate the mean density of the earth, which he found, by his first calculation, to be 4.56, the density of water being taken as the unit; but which, on a careful revision of the process, he has increased to 5.

Now as the surface of a large part of the globe is covered with water, and the density of the earthy matter, that forms by far the gratest portion of the residue, is little more than twice as great as water, it becomes evident, that the earth is denser within than at the surface; and as we can detect no sudden increase of density, it is probable that the variation in this respect is regular.

Observations on the attraction of mountains, made in the neighbourhood of Marseilles, by the Baron de Zach, give analogous results; and we shall hereafter cite experiments. made with the pendulum, that corroborate their accuracy.

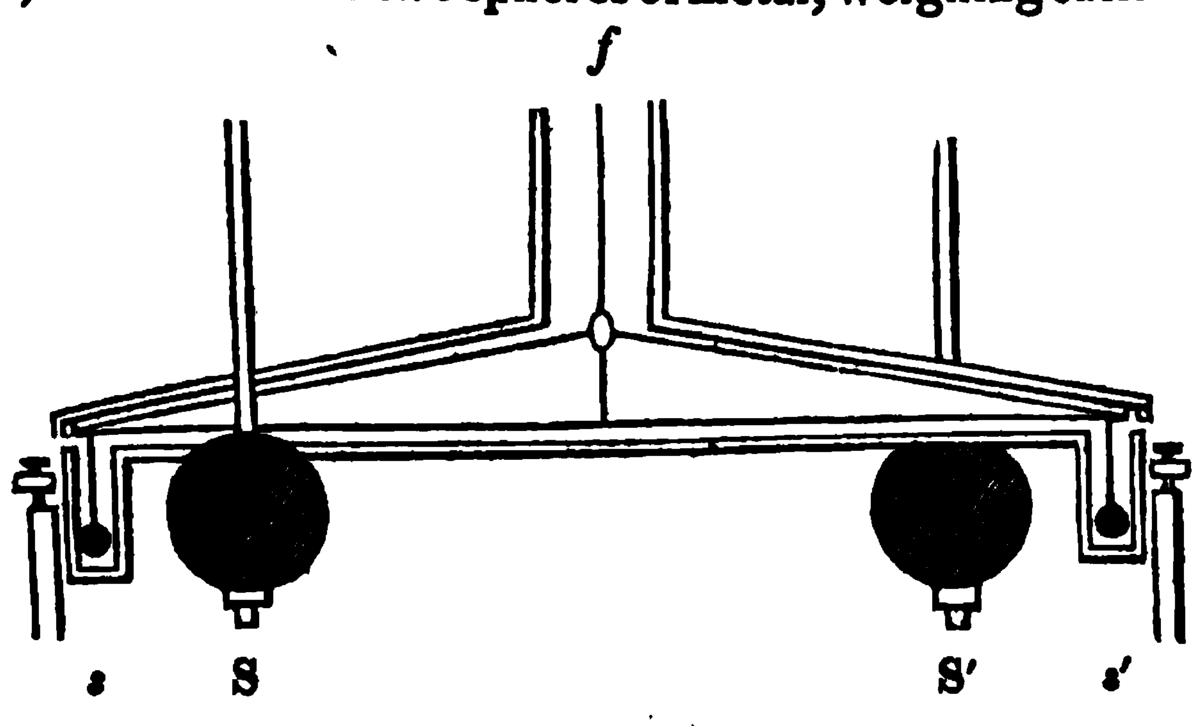
90. The mutual attraction of bodies near the surface of the earth, has been detected by Cavendish, and his observations have furnished another determination of the mean density of the earth. The apparatus employed by him is admirably suited to the purpose for which it was intended. It was originally planned by

Mitchell, a member of the Royal Society of London. This experimenter being prevented by illness, which proved fatal, from completing his researches, left by will his apparatus to Francis I. H. Wollaston, and from him it passed into the hands of Cavendish. The principle of this apparatus may be explained as follows:

If a bar of an inflexible substance be accurately poised by its middle, in a horizontal position, by means of a thread or wire, the nature of the thread or wire is such as to bring it to rest in one particular position. A small force will be sufficient to withdraw the bar from this position, but the twisting or torsion which this deflection will cause in the wire, will gradually oppose an increasing resistance, until this latter exceed the deflecting force; the torsion will then cause the bar to return to its original position, whence the deflecting force will again compel it to move. Hence the bar will oscillate between two points, determined by the intensity of the deflecting force, and that of the torsion of the wire. The rapidity of the oscillations will, upon principles that we shall hereafter explain, furnish a measure of the intensity of the deflecting force.

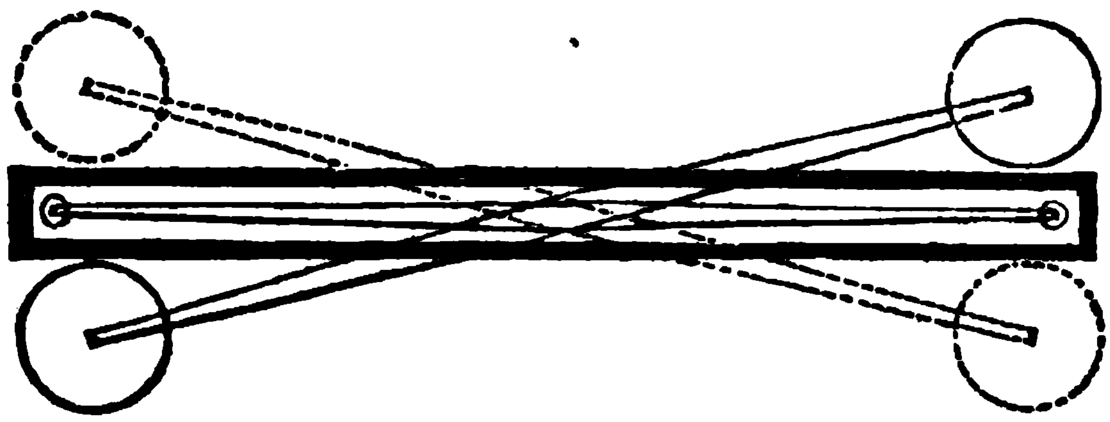
Now if bodies mutually attract each other, a considerable mass of a heavy substance, of a metal for instance, placed in the same horizontal plane with the bar, ought to cause a deflection, and consequent oscillation; for, in this position, the attractive force of the Earth is completely counteracted by the exact poising of the bar; hence that influence, which would in most other cases cloak, and render imperceptible, a force so much smaller as that exerted by the heaviest masses we have it in our power to move, would become neutralized.

91. Such being the principle, we shall proceed to describe the apparatus more in detail. It is represented in the figures annexed; S and S′ are the two spheres of metal, weighing each 350 lbs;



a box is represented in which the bar is shut up, in order to preserve it from the action of currents of air ; *s* and *s'* are two small balls suspended from the extremities of the moveable bar, and by whose weight it is kept exactly balanced and in equilibrio.

The following figure is a horizontal section of the same apparatus.



In the previous figure, will be seen the manner in which the two small balls are suspended from the bar by a silver wire ; this wire passes through the bar, and is attached to the vertical wire, *f*; the tenacity of the latter is just sufficient to bear, without risk of breaking, the weight of the bar and the two balls *ss'*, and its torsion is the only force that opposes their oscillation. The two masses S and S', are themselves supported by iron rods, attached to a horizontal arm in such a manner as to have each a free motion in a semicircular arc, one on each side of the box: they can be placed in either of the positions represented in the second figure, by an operation that is performed on the outside of the chamber in which the apparatus is enclosed. This chamber had neither doors nor windows, and was illuminated by means of a lamp, which was placed without the chamber, in order that the interior might not be heated. An aperture opposite to the place of the lamp, admitted the light to fall upon one end of the bar, and the oscillations were noted by means of a small telescope, passed through the wall, just beneath this opening.

The whole apparatus being at rest, and the masses SS' in a position at right angles to the bar, they were turned around until they reached the position represented in the second figure. As soon as this was done, the bar began to move, and the balls *ss'* to oscillate.

The amount of these oscillations furnishes a measure of the attraction of the masses SS', at a given distance, and this being known, and compared with the effects of the attraction of gravitation, exerted by the earth, a simple statement in proportion will give the mass of the latter. Its density was hence calculated by Cavendish to be 5.48.

Hutton has detected an inacuracy in the calculations of Cavendish, by correting which the result has been lessened, and brought more near to the inference drawn from the experiments of Schehallien.

Still, the difference is greater, between the results of the two methods, than it ought to be. The same distinguished mathematician has, therefore, proposed that an experiment similar to that of Schehallien, be repeated on the opposite faces of the great Egyptian pyramid, at the height of one fourth from the base. The regularity of the figure of this great artificial mass, and its probable identity of composition throughout, render this method so accurate, as in all probability to settle this disputed question. Until this be done, we may without much risk of error, assume that the mean density of the earth is five times as great as that of water.

92. Knowing the density of the earth when compared with water as the unit, the weight of a given bulk of water, and the volume of the earth, we may proceed to calculate its weight, in the units of a conventional system of measures. Astronomic observations, and the calculations founded upon them, enable us to compare the mass of the earth with that of the sun and moon; knowing thus the mass of the sun, we calculate readily the mass of any planet accompanied by a satellite; and the determination of the mass of those planets that have no satellites, although less easy, is also practicable; thus, as has been well remarked, the apparatus of Cavendish is in fact a balance, in which we can weigh the vast bodies that compose the solar system.

93. As bodies fall to the earth when left without support, at all places within our reach, it may be inferred that the attraction of gravitation acts during the whole time of their descent, and is therefore an accelerating force. This is also conclusively shown, by the well known fact of the acceleration that takes place in the motion of falling bodies; these strike the earth with greater velocity, in proportion as the distance through which they have fallen increases. We cannot however, have recourse to experiments of this kind, in order to discover whether this accelerating force be constant, or variable in its intensity; for, as the air causes an unequal velocity in bodies of unequal densities, and prevents the fall of some altogether, and as we cannot obtain a vacuum of sufficient extent for the experiment, we cannot receive the indications obtained by the fall, even of the densest bodies, as absolutely accurate.

The resistance of the air, as will be hereafter seen, varies with the square of the velocity; hence, if the velocity of a falling body be diminished in an arithmetic progression, the resistance of the air decreases in geometric ratio. Could we therefore observe the motion of a body actuated by gravity, under circumstances where its motion would be diminished, without any alteration taking place in the law of the accelerating force, we might obtain

that law, almost without its being affected by the resistance of the air.

94. We may consider the force of gravity as acting parallel to itself within small horizontal distances; a body therefore, moving upon a plane slightly inclined to the horizon, will suit our purpose; for, by §56, the force by which a body is actuated, when it moves on a plane, inclined to the direction of an accelerating force at a constant angle, follows the same law with the accelerating force itself. That is to say, that although the absolute velocity is lessened, the relation between the spaces described, during different portions of the time of the motion's continuance, is constant. Hence, should we find that the velocity with which a body descends on an inclined plane is uniformly accelerated, we may infer, that the force of gravity is constant, within the limits of the plane's elevation.

Experiments were made by Galileo upon this principle. He formed his inclined plane by stretching a cord twenty or thirty feet in length, between the two fixed points, at different levels. The difference of height between the two points must be small, and the inclination of the plane is therefore small also; the velocities were diminished, according to the principle in §56, in the ratio of the sine of the plane's inclination to the horizon. In order to lessen the friction, the bodies were mounted upon wheels, and the smallness of the velocities rendered the resistance of the air insensible. Experiments made with this apparatus, showed an uniform acceleration in the velocity of the bodies; and hence, that gravity was, within the limits occupied by the plane, a constant accelerating force.

95. The machine of Atwood affords a more convenient and elegant method of obtaining the same results.

Two bodies, of unequal weights, are united by a cord passing over a pulley. It is therefore obvious that the preponderance of the one will cause it to descend, and raise the other, through an equal space; but it is also obvious, that this motion cannot be as rapid, as that which either body would have, if free. The relation which the accelerating force, in such a system, bears to the whole force of gravity may be easily investigated.

Let A and B be the two bodies, and g the force of gravity, which is the accelerating force that acts upon them both, but whose effects are modified by their mutual action.

Let t be the time elapsed since the motion began; v the velocity of A; v' the velocity of B.

These velocities are obviously equal, but in contrary directions.

During the time dt, the velocities will increase under the action of the accelerating force, by quantities which will be represented by dv and dv'; during the same time the bodies, if falling freely,

would each acquire (§ 49) the velocity gdt. Hence the body A will lose, in consequence of its connexion with B, a velocity represented by $gdt—dv$; and B will lose by its connexion with A, a velocity represented by $gdt—dv'$. These are respectively equal to a'' and b'' in the formula (98), $Aa''+Bb''+\&c.=0$, that expresses the principle of D'Alembert, while A and B are respectively equal to the quantities of the same name; and the motion of the bodies being contrary, the second term becomes negative; therefore, by substitution, we have

$$A(gdt-dv)-B(gdt-dv')=0;$$

and as the velocities are equal, but with contrary signs,

$$dv=dv';$$

therefore,

$$A(gdt-dv)-B(gdt+dv)=0;$$

performing the multiplications, we obtain

$$Agdt-Bgdt-Adv-Bdv=0;$$

by transposition,

$$(A-B)gdt=(A+B)dv;$$

dividing by A+B,

$$dv=\frac{A-B}{A+B}gdt; \qquad (104a)$$

integrating,

$$v=\frac{A-B}{A+B}gt+a;$$

in this expression the arbitrary constant, a, represents the initial velocity, which when the bodies move from rest, becomes equal to 0; in which case

$$v=\frac{A-B}{A+B}gt. \qquad (104b)$$

The forces which impel equal moving bodies, or systems of bodies, being proportioned to the velocities, we obtain the following law.

The force which remains to cause the descent of the heavier body, in Atwood's machine, is to the whole force of gravity, as the difference of the weights of the two bodies is to their sum.

Experiments with this apparatus, show a motion uniformly accelerated; hence we have a farther proof that the attraction of gravitation is a constant force, within the space occupied by the machine.

As the formula we have investigated, gives the relation between the velocity the body A has, when united to B, and that which it would have when falling freely, this machine gives us a ready mode of obtaining the velocity of falling bodies. Thus if the body A have a weight of 101, and B of 99, their sum will be 200, their difference 2, and the proportion of the actual velocity, to that obtained by falling freely $\frac{1}{100}$. Now a velocity no more

than $\frac{1}{10}$th part of that acquired by a falling body, is so slow, that it will be perfectly easy to note the marks upon a vertical scale attached to the instrument with which the descending weight corresponds, at each beat of the clock that is also attached. By such experiments, carefully conducted, it is found that the actual descent, in the instance we have stated, is 1.93 inches in the first second; whence the fall of a heavy body from rest may be inferred to be 193 in. in the first second of time, or 16 feet and an inch.

The machine of Atwood is represented on the opposite page.

In this will be seen,

1. The pulley, *a*, the extremities of whose axles each rest upon two other wheels, *bb*, the axles of which rest on agate planes; in this manner, as will hereafter be shown, the friction is reduced to the smallest possible.

2. A divided scale along which the body A descends without touching it. On this scale are placed two moveable plates *f* and *g*, one pierced in the form of a ring, the other a plane surface. The weights are at first equal, and the preponderance of A is effected, by laying upon it a body of the shape of a bar, which cannot pass through the ring; thus the accelerating force may be removed at any point, and the velocity will become uniform; whence not only the spaces described by the accelerated motion, but the final velocities also, may be made the subjects of experiment.

3. In order to count the time, during which the motion takes place, a clock beating seconds is attached to the side of the machine, and a spring is adapted, in such a way, that the weight A, and the pendulum P, of the clock E, may be released at the same instant of time.

Applied to ascertain the final velocities, this machine gives the results obtained from theory in § 56, namely, that the velocities acquired by moving from rest, under the action of an accelerating force, are such as to carry the body with uniform motion through twice the space, in a time equal to that employed in acquiring those velocities.

96. The inferences obtained from the experiments made with the inclined plane of Galileo, and the machine of Atwood, have been purposely restricted to the limits occupied by the respective apparatus; for it will, even at first sight, appear probable, that the force of gravity, residing in the mass of the earth, although apparently constant within small limits, must decrease as we recede from the earth, according to some law dependent upon, or in mathematical terms, according to some function of the distance.

Galileo, who first ascribed the fall of heavy bodies to a mechanical force exerted upon them by the earth, did not attempt to


d
d'

extend the action of that force to bodies not in the immediate vicinity of the earth. Kepler, with more extended views, saw that it could not be thus limited in its action, but failed in discovering the law, according to which its intensity varies, in terms of the distance. His views were in consequence neglected, and almost completely forgotten.

Newton, like Kepler, saw that the action of gravity, which does not diminish in a degree that it had been possible previous to his time to detect, even at the top of the highest mountains, could not cease suddenly. He hence inferred that it might extend to the moon, and be the cause by which that body was compelled to describe a re-entering orbit. But although it might extend to the moon, it was improbable that it could act there, with an intensity equal to that it exerts, at the surface of the earth. The directions of gravity, supposing the earth to be spherical, are directed to the centre: hence the influence, whatever be its cause, is a radiating one, and would probably follow the same law in its decrease, as the radiating actions of light and heat. The force of these natural agents, is well known to vary, in the inverse ratio of the squares of the distances from the centre. Newton inferred, that the decrease in the intensity of gravity, must follow the same law; and that at the moon, the attractive force must be as much less than it is at the surface of the earth, as the square of the radius of the earth is less than the square of the moon's distance from the earth's centre.

The moon, if not acted upon by a force drawing it to the earth, would, according to the principles of motion, § 62, go off in the direction of a tangent. Taking the motion of the moon for a short space of time, and consequently in a small arc, the tangent of the arc will represent the force with which the moon would proceed in a rectilineal direction, the space between the tangent and the extremity of the arc, will represent the measure of the deflecting force. If the arc be taken which corresponds to the mean motion of the moon for a minute of time, this space may be calculated to be about sixteen feet. Now the measure of the force of gravity, g, at the surface of the earth is twice the space a body falls from a state of rest in a second of time, or is in the nearest round numbers 32 feet. The mean distance of the moon is about 60 semidiameters of the earth, and the respective distances of the two bodies have the ratio of 60 to 1. assuming with Newton, that the force varies, in the inverse ratio of the squares of the distances, the value of g', the force of gravity at the moon, is

$$g' = \frac{32}{(60)^2};$$

the space described in a given time, under the action of a constant accelerating force, and within this small space the force may be considered constant, is by the formula (61)

$$s=\frac{gt^2}{2};$$

taking the value of g' given above as the measure of g the accelerating force in this formula, and making

$$t=60'',$$

or a minute of time, we have

$$s=\frac{32+(60)^2}{2+(60)^2}=16 \text{ feet.}$$

The deflection of the moon from a tangent to her orbit in a minute of time, is therefore the same with that which it should be, were she acted upon by an attracting force residing in the centre of the earth, and varying in its intensity, in the inverse ratio of the squares of the distances.

By an investigation analogous to this, Newton inferred that the attractive force of the earth did not cease suddenly, but extended as far as the orbit of the moon, and acted there according to the law observed in other forces, or influences, emanating from a centre.

97. The experiments that have been cited § 91, showing a mutual attraction between gravitating bodies, it may readily be inferred, that the moon has an action upon the earth proportioned to its mass. This attraction of the moon may be shown by its influence upon the liquid mass of the ocean, in which it forms waves, that constitute an important part of the tides: it is also obvious in various astronomic phenomena that are foreign to our subject.

98. It being thus found that the attraction is mutual between the earth and moon, analogy leads us to infer that the earth and sun mutually attract each other, with forces proportioned to their respective masses, and that the same mutual action takes place between all the bodies that compose the solar system. The astronomy of observation fully confirms this important truth; and physical astronomy, thus established on a secure basis, applies the principles of mechanics to investigate the minute changes and variations that this multitude of forces, exerted by moving bodies, are constantly producing in each other's motions. Thus the simple mechanical laws, which we are compelled to investigate in order to explain the action of the most familiar machines, are the same that direct the vast mechanism of the heavens. Nor does the research that these laws give rise to, stop at the bounds of our own system. The fixed stars also obey the same universal force; and the astronomers of Europe are now on the threshold of discoveries in respect to them, more wonderful than even those with which Laplace has closed the labour began by Newton, leaving no motion in our planetary system, that is not reducible to mechanical principles.

99. The law according to which gravity decreases, shows us, that although to all appearance constant within the limits of our experiments, it is not absolutely so; and we have at present modes of observation that will hereafter be noticed, by which even the small decrease that occurs in distances within our reach, may be rendered evident.

100. Were the earth at rest in space, and perfectly spherical in its form, the force of gravity would be constant at every point of its surface; but if it be in motion this cannot be the case.

When a body revolves around a fixed axis, its several points describe circles whose planes are perpendicular to, and their centres in, the axis. From § 64, it would appear that these points, revolving in equal times, are influenced by centrifugal forces that are proportioned to the radius of the circles they respectively describe; and the directions of the forces, are in the planes of these circles. Now these forces must lessen the action of the force of gravity, which is due to the attraction of the whole mass of the earth, but they will affect it differently in different latitudes. At the poles, the centrifugal force is equal to 0, for the circle becomes a mere point; while it is greatest in points situated in the equator, as that is the greatest circle of diurnal rotation. At the equator, too, the centrifugal force acts in direct opposition to the force of gravity, while in all other places they are inclined to each other.

The law which the decrease of gravity, influenced by these two circumstances, follows, upon the surface of the earth, between the poles and the equator, may be thus investigated:

The general expression for the centrifugal force is (83)

$$f=\frac{4\pi^2 r}{T^2};$$

call the diminution of gravity at any latitude f';
the radius of the circle, described by any point on the surface, is equal to the cosine of the latitude; the time T is a sidereal day, and is constant; substituting the former value, and calling the latitude L, we have

$$f'=\frac{4\pi^2 \cos. L}{T^2};$$

but this force does not act in direct opposition to the force of gravity, the one being parallel to the equator, while the other is directed to the centre. We must therefore resolve the centrifugal force f' into two others, one of which is parallel, and the other perpendicular to the surface of the earth, at the given latitude; the latter is that which acts to diminish the force of gravity, and may be found by the formula of the Resolution of forces (11)

$$X=R \cos. a,$$

hence the expression for the value of the centrifugal force must be multiplied by the cosine of the angle its direction makes with the plumb-line, or by the cosine of the latitude, and the value of the diminution of gravity, at any latitude, becomes

$$\frac{4\,\pi^2 \cos^2 L}{T^2}. \qquad (105)$$

If then the earth were a perfect sphere, the centrifugal force at the equator would bear to the force that lessens the attraction of gravity at any latitude, the ratio of the radius of the earth to the square of the cosine of the latitude.

101. The ratio of the centrifugal force at the equator, to the whole force of gravity, may be ascertained in the following manner:

Let G be the whole attractive force of the Earth at the surface, g the apparent force of gravity at the Equator, then

$$g = G - \frac{4\pi^2 r}{T^2}. \qquad (106)$$

In this expression, G is twice the space a body descends from rest in a second of time, or

$$G = 32.16 \text{ feet.}$$

and, $\pi = 3.1416$;

r is the Earth's radius, which calculated in feet, gives

$$r = 41{,}836420 \text{ feet};$$

the sidereal day reduced to seconds gives

$$T = 86164'';$$

substituting these values,

$$g = 32.16 - 0.1118; \qquad (107)$$

and as

$$\frac{32.16}{0.1118} = 288,$$

therefore—

The ratio of the centrifugal force at the equator is to the apparent force of gravity as 1 to 288; or the diminution of gravity, between the poles and the equator, is $\frac{1}{288}$th part of the whole.

102. These relations have been investigated upon the hypothesis of the earth's being a sphere, but this could only occur in a moving body, in the case of its being a solid mass devoid of all elasticity. Every point in a moving spherical body, being influenced by a centrifugal force, greatest at the Equator and least at the Poles, these points would tend to assume a state of equi-

librium, under the joint action of the centrifugal and gravitating forces. If then the mass were originally spherical, the equatorial parts would tend to recede from, the polar to approach the centre; and were they free to move, the equatorial diameter would be increased, and the polar would be diminished, until a state of equilibrium were obtained. Now although we know nothing certain, in respect to the state in which the interior of the earth exists, and find its outer crust a solid body, yet in large basins or cavities of that crust, a fluid, (the ocean,) exists, covering nearly three-fourths of the surface. The level of this mass of fluid points out, as has been seen, the mean surface of the earth. Now this fluid mass could only be in a permanent state, if the general shape of the crust had the form the fluid would itself assume. Hence in whatever state the mass of the Earth may have been originally created, its shape is that of a fluid body retained in equilibrio, by the joint action of the centrifugal force and that of gravity. This form has been investigated under a variety of hypotheses, and by different persons. It is not our province to enter into these investigations; we shall therefore content ourselves with simply stating a few of the results.

Newton, considering the earth as a homogeneous fluid mass, endued with a rotary motion, and composed of particles, attracting each other, according to his law of universal gravitation, took it for granted, that it would assume the figure of an oblate spheroid. With these data, he inferred that the flattening of such a spheroid would be $\frac{5}{4}$ths of the relation of the centrifugal force at the equator, to the whole force of gravity. The latter has been shown to be $\frac{1}{289}$, and hence the ratio between the equatorial and polar axes would be, upon his hypothesis, 230 : 229.

Huygens, assuming the whole force to reside in the centre, determined the ratio to be 578 : 577.

The last result has been shown since, to be consistent with the theory of mutual attraction, under the hypothesis, that the earth is infinitely dense at the centre, and infinitely rare at the surface. Now as we have shown, (§ 91) that the mean density of the earth is greater than that of its surface, and hence inferred, that the density increases towards the centre, the figure of the earth must vary between these limits, and the oblateness of the terrestrial spheroid cannot be greater than $\frac{1}{230}$ nor less than $\frac{1}{578}$.

Such being the theory, it will be obvious, that if a flattening at the poles can in any manner be detected, such flattening would furnish conclusive evidence of the diurnal motion of the earth. The same would be shown by a decrease in the apparent action of gravity, between the poles and the equator.

Both of these may be ascertained to exist, by methods that remain to be described, but the oblateness may and has been detected by actual measurement.

103. To recapitulate the laws of Universal Gravitation.

(1.) It is common to all bodies, and mutual between them.

(2.) It is proportioned to the quantity of matter in the body.

(3.) Its intensity decreases, as the square of the distance from the centre of attraction increases.

CHAPTER III.

Of the Centres of Gravity and Inertia.

104. From what has been said in the last chapter, on the subject of the Attraction of Gravitation, it appears, that bodies near the surface of the earth, being, like all others, influenced by it, may be considered as acted upon by a number of gravitating forces, tending to draw each particle, or material point, in their mass, towards the centre. These forces, within the space occupied by even the largest bodies, may be considered as equal; and although their directions actually converge, yet in any given body the convergence is insensible; no error can therefore arise from considering them as absolutely parallel.

A heavy body, near the surface of the earth, may therefore be considered as acted upon by a number of equal and parallel forces. These forces have a resultant, which is equal to their sum, and is identical with the *weight* of the body. This weight will depend upon the volume of the body, its density, and the intensity of gravity at the place in which it is situated. If we call the volume or bulk B, the density D, and the measure of the force of gravity g, the value of the weight W will be

$$W = B\,D\,g,$$

and as by (101) the mass, $M = B\,D$,

$$W = M\,g.$$

105. The point of application of this resultant, which, in the abstract theory of parallel forces, has been called their Centre, is in the case of gravitating bodies called the Centre of Gravity.

The formulæ then of § 16, by means of which the centre of parallel forces is found, and the several inferences obtained from those formulæ in particular cases, are applicable to the subject before us. So also are the inferences from the geometric investigations, in the same section. That this is true in respect to homogeneous solid bodies is evident, for they may be considered as made up of a number of equal particles, uniformly distributed throughout the mass. In practice, however, it frequently becomes necessary to determine the centres of gravity of lines and surfaces, and even of abstract figures of three dimensions. For this purpose, they are supposed to be divided into an infinite number of small and equal parts, each of which is influenced by an equal gravitating force.

The inferences before obtained, may be now recapitulated, in reference to this individual case of parallel forces.

(1.) The centre of gravity of a straight line bisects it.

(2.) The centre of gravity of two straight lines is found by joining the two points that bisect them, and dividing the line that joins them, into parts reciprocally proportional to the magnitudes of the two lines. Of three lines, the centre of gravity may be found, by first finding the centre of gravity of two of them; this is then to be joined by a straight line to the point that bisects the third, and this last line divided into parts reciprocally proportioned to the joint magnitude of the two first, and the third line. The centre of gravity of four lines may be found, by first finding the centre of three of them, and uniting it to the point that bisects the fourth, which is then to be divided as in the former case. In this manner the centre of gravity of the perimeter of a triangle, or other figure, bounded by straight lines, may be determined.

(3.) The centre of gravity of the surface of a triangle, is in the line that joins the vertex to the point that bisects the base, at the distance of two-thirds of that line from the vertex.

(4.) The centre of gravity of a quadrilateral figure may be found, by dividing it into two triangles, joining their respective centres of gravity, and dividing the line that unites them into parts reciprocally proportional to the area of the two triangles. And in general the centre of gravity of any polygon whatever may be found by dividing it into triangles, the centre of gravity of two of which is first found, and united to the centre of gravity of the third by a line, that is to be divided in a similar ratio; this point is then to be united by a straight line, to the centre of gravity of the fourth triangle, and so on. Any polygon, it is well known, can be divided into as many triangles, less two, as it has sides.

(5.) The centre of gravity of a circle is in its centre. This point is also the centre of gravity of a ring contained between two concentric circles.

The centre of gravity of a circular arc, is at a distance from the centre of the circle, which is a fourth proportional to the lengths of the arc, the chord, and the radius of the circle.

In a semicircle this distance is

$$\frac{r}{1.5708}=0.63662\,r.$$

(6.) The centre of gravity of a parallelogram is in the point where its two diagonals intersect each other.

(7.) The centre of gravity of a parabola is in its axis, at the distance of three-fifths of that line from the vertex.

(8.) The centre of gravity of the surface of a solid, formed by the revolution of a plane surface around a line, in respect to which, all its parts are symmetrically situated, is the same as the centre of gravity of the generating surface.

Thus: the centre of gravity of a hollow cylinder is in the point that bisects its axis; the centre of gravity of a hollow cone is in its axis at a distance of two-thirds the length of that axis from the vertex. The centres of gravity of hollow spheres, and ellipsoids of revolution, are in their centres of magnitude.

(9.) The centre of gravity of the surface of a spheric segment bisects its versed sine.

(10.) The centre of gravity of a triangular pyramid is in the line that joins its vertex to the centre of gravity of the base, and at the distance of three-fourths of that line from the vertex.

(11.) The centre of gravity of any solid figure, bounded by plane surfaces may be found by dividing it into a number of triangular pyramids, and proceeding in them as has been directed to be done in regard to the triangles, into which plane surfaces are divided for a similar purpose.

(12.) The centre of gravity of a solid cone, is in its axis at the distance of three-fourths of that line from the vertex.

(13.) The centre of gravity of a sphere is in its centre of magnitude, as is the centre of gravity of a shell contained between two concentric spheres.

(14.) The centre of a gravity of a solid paraboloid is at a distance from the vertex, equal to two-thirds of the axis.

(15.) The centre of gravity of a spheric segment is in its fixed axis, at the distance of $\frac{5}{8}$ths of its length from the vertex.

(16.) The centre of gravity of a cycloid, that is bisected by the vertex, is in the diameter of the generating circle, at a distance of one-third of the perpendicular height from the vertical arc.

106. In order that a heavy body shall be supported, it is necessary that a force shall be applied to it, equal in magnitude, and contrary in direction, to the force of gravity that acts upon it. The direction of gravity is a vertical line, and its point of application is the centre of gravity; hence the supporting force must act perpendicularly upwards, and must be applied either to the centre of gravity itself, or somewhere in the vertical line passing through it, which is called its Line of Direction.

If the supporting force be applied to a single point in the body, there are three cases that may occur:

(1.) In the first place, the point of support may be above the centre of gravity; in this case, the centre of gravity will be in the vertical line passing through, or will be directly beneath, the point of support; this case occurs in all instances of suspension, where

the line of support is vertical, and the centre of gravity is in the line of support produced; when this occurs, the centre of gravity is in the lowest possible point. If the equilibrium of such a system be in any manner disturbed, the body will oscillate on each side of the vertical line, and if, as always happens in nature, its motion be opposed by resistances, the body speedily returns to rest in its original position; hence the equilibrium is said to be stable.

(2.) The supporting force may be applied exactly to the centre of gravity. In this case, if the body be moved from its original position, the forces have their direction changed in respect to any line taken in the body, and supposed to be at rest; it is therefore an instance of that case in parallel forces, where the forces revolve around their points of application, without ceasing to be parallel; for the result will be the same, whether the forces themselves move, or the points of application turn around the centre of force. Hence, a body supported by its centre of gravity, will remain at rest in any position in which it is placed; the equilibrium is now said to be indifferent, as the body has no greater tendency to remain in any one position than in another.

(3.) The point of support may be below the centre of gravity; in this case, as the opposite directions of the supporting and gravitating forces must coincide in the same vertical line, the centre of gravity will be immediately above the point of support. If the equilibrium be disturbed, the centre of gravity must describe a circle around the point of support, hence the centre of gravity is in the highest possible point; and as this motion is in a curve concave to the horizon, the motion will continue around the point in the same direction as at first, until the centre of gravity come immediately beneath the point of support, or until it meet some new point of support, by means of which the centre of gravity may be sustained in a state of stable equilibrium. As the body can never return of itself to its original position, the equilibrium, when it is supported from beneath, is said to be tottering, or unstable.

All feats of balancing depend upon these properties of the centre of gravity. They, generally speaking, consist in a skilful application of a small force to retain the body in its position of tottering equilibrium, even after the conditions are partially disturbed. Sometimes the point of support is fixed; the art then consists in changing the distribution of the weight, in such a manner as to bring back the line of direction of the centre of gravity to the point of support. Sometimes the point of support is moveable, and the skill is then shown, by changing its position in such a manner, as to make the line of direction of the centre

of gravity constantly move its position, so as to meet, and pass through the point of support.

107. A body may be supported upon a sharp edge. In this case, the centre of gravity will be in the vertical plane passing through the edge: and here again the equilibrium may be stable, indifferent, or tottering, according as the centre of gravity lies below, in, or above the line, which marks the meeting of the two surfaces that form the edge.

108. The body may rest upon a surface. In this case, equilibrium can only occur when the line of direction of the centre of gravity falls within the base on which the body stands. Although the supporting force is a normal to the surface, and in order that equilibrium may exist theoretically, the surface of support ought to be parallel to the horizon; still there are certain forces that act to prevent a body from sliding, even upon an inclined surface. Such forces will be hereafter examined and described. It is sufficient for the present to state, that a body may be in equilibrio upon a base of small inclination. This however can only be the case, when, as in the former instance, the line of direction of the centre of gravity falls within the base.

109. If the body be of such a form as to touch the horizontal plane, on which it rests, only at a single point, the three several species of equilibrium may exist, according to the form of the body, and its position in respect to the plane.

If from the centre of gravity of the body, lines be supposed to be drawn to every point of its surface, some of the lines will be always normals to the surface, while others will be oblique. If the body rest upon any of the points through which one of these normals passes, equilibrium will take place; if this normal be the shortest line that can be drawn from the centre of gravity to the surface, the equilibrium is stable, for the centre of gravity will be directly above the point of support, and will also be in the lowest possible position; hence any disturbing force will only cause an oscillation in the body, which will finally return to rest in its original position. The same is the case if the normal, although not absolutely the shortest line drawn from the centre of gravity to the surface, is relatively shorter than those contiguous to it. If the body rest upon the point where the longest normal intersects the surface, the body is in a state of tottering equilibrium, and will, if disturbed, turn around until it rest upon the shortest normal; the same will occur, even if the normal be not absolutely the longest line, but if it be longer than the other lines, drawn from the centre of gravity to the surface, which are in its immediate vicinity. If all the lines drawn from

the centre to the surface be normals, the equilibrium will be indifferent, or if the normal on which it rests be situated in the vicinity of other lines that are also normals.

As instances:

A portion of a homogeneous sphere, or of a spherical surface equal to, or less than, a hemisphere, will have stable equilibrium; the same will take place in an ellipsoid formed by the revolution of an ellipse around its shorter axis, which will come to rest with that axis in a vertical position.

A homogeneous sphere will remain in any position in which it is placed upon a plane, and is hence in a state of indifference.

An egg, or an oblong ellipsoid of revolution, will be in a state of tottering equilibrium, if poised upon its longer axis; while if laid on one side, as all the radii of the circular section, in which the points of contact are situated, pass through the centre of gravity, the equilibrium is indifferent.

A portion of a cylinder not greater than the half, cut off by a plane parallel to its axis, and laid on the curved surface, comes to rest upon the line, in which a plane perpendicular to that by which it is cut from the cylinder intersects the surface, and is therefore in a state of stable equilibrium.

These circumstances may be illustrated by the following figures.

Fig. 1

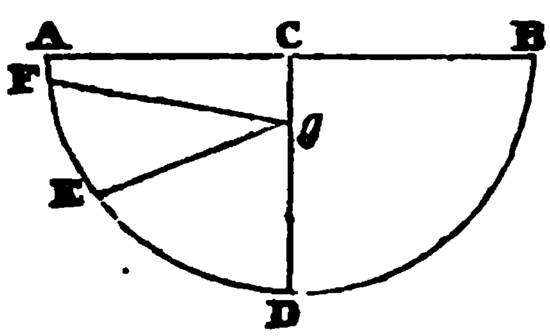
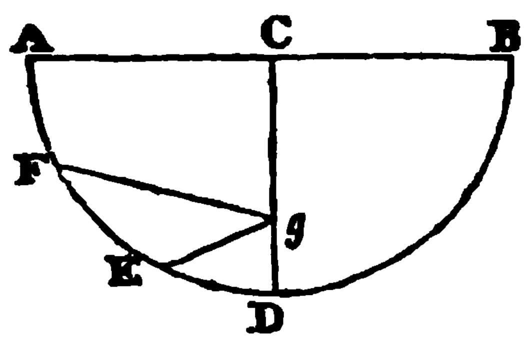


Fig. 2.



Let the above figures represent bodies whose sections are semicircular, Fig. 1 being solid, Fig. 2 being hollow. In either case, the normal *g*D will be the shortest line that can be drawn from the centre of gravity, *g*, to the curved surface. If any disturbing force act, that is not sufficient to bring the body into the position, in which lines passing through *g*, and A or B, are vertical, the body will finally return to rest in the position in which *g*D is vertical: when the disturbing force is removed, it will oscillate in returning to rest, until the resistances overcome its motion. In the case of a spherical surface, the oscillations may take place in any direction whatsoever, but in the case of a portion of a cylinder, only in planes parallel to the circular section. In a solid spheric segment, the distance C*g* is only $\frac{3}{8}$ths of CD; while in a hollow spheric segment, the centre of gravity will coincide

with that of the curve, and its distance from C will be nearly $\frac{2}{3}$ds of CD. A similar difference in position will take place, in relation to portions of solid, and of hollow cylinders. Hollow bodies of these classes, are, therefore, more stable than solid ones; and with equal weights, will more powerfully resist any effort to disturb their equilibrium.

In the homogeneous sphere, one of whose great circles is represented by the circle ADBE, Fig. 1, the centre of gravity,

Fig. 1.

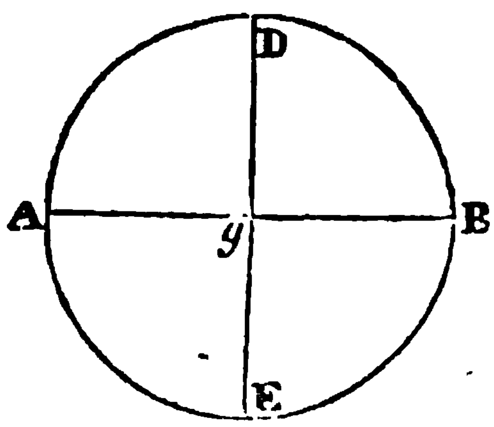
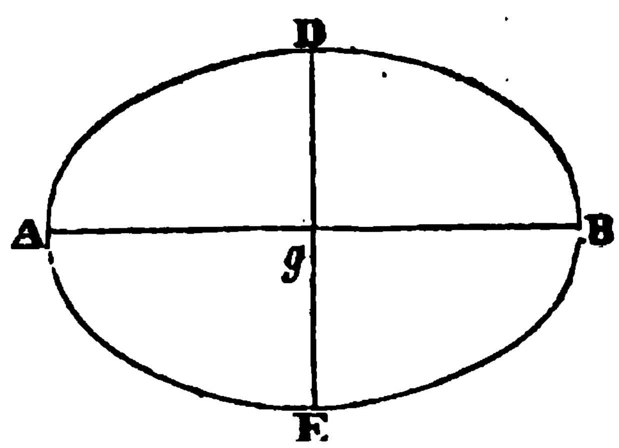


Fig. 2.



corresponds with the centre of magnitude, and the lines of direction will be all normals, and of equal length, upon whatever point it rest; hence its equilibrium is that of indifference. If the figure represent the circular section of a cylinder, a similar state of indifference will exist in one direction. If in Fig. 2, the elliptical curve ADBE, represent the section of an ellipsoid, formed by the revolution of the curve upon its shorter axis, and the solid thus formed be homogeneous, the centre of gravity will be in the centre of magnitude *g*. The lines of direction *g*D and *g*E, which are normals to the surface, are the shortest that can be drawn within the body, it will therefore have, when resting upon either of the points D or E, a state of stable equilibrium. All lines in the plane of *g*A, and *g*B, are also normals to the surface, but are the longest lines of direction that can be drawn within the body: hence, if resting upon the points A or B, it will be in a state of tottering equilibrium, in case the disturbing force act in the plane of AB. If the same curve represent the section of an ellipsoid, formed by revolution around its longer axis AB, the lines *g*A and *g*B, are normals, and the two longest lines of direction that can be drawn within the body, resting on the points A and B; therefore, its equilibrium is tottering. All the lines of direction that can be drawn in the plane of DE, are equal among themselves, and shorter than any other lines that can be drawn within the body from the point *g*; hence in respect to forces acting in the plane of DE, the equilibrium is indifferent.

110. If the surface of a body resting on a plane, be also a plane, and the line of direction of the centre of gravity fall within it, the equilibrium is of course stable. If extrinsic forces act to disturb the position of the body, the more extensive the

plane surface on which it rests, the nearer to the centre of the surface the line of direction falls, and the lower the position of the centre of gravity, the more the body will resist a force applied to overturn it. When the force that acts to overturn it is sufficient for the purpose, the body, if not broken by its action, will turn around one of its solid angles as a centre, or round one of its edges as an axis. The centre of gravity must of course rise in a circular arc, and with it the weight of the body; the resistance of the body to the effort to overturn it, will therefore depend, not only upon its own weight, but upon the position and curvature of the arc described by the centre of gravity. Some of the cases that may occur in practice, are represented below.

The triangle ABC, Fig. 1, is the section of a pyramid or cone whose centre of gravity, *g*, is at the distance of $\frac{3}{4}$ths of its

Fig. 1.

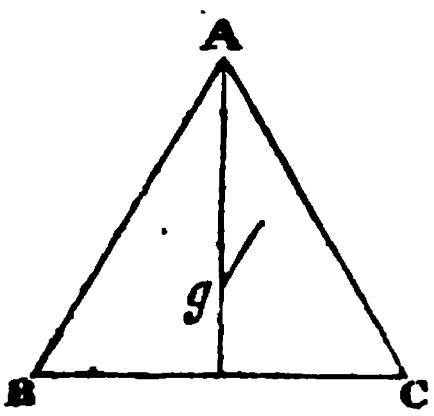
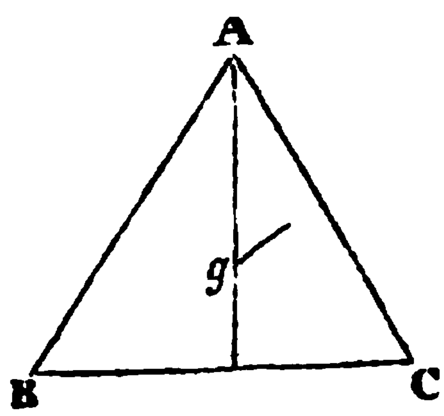


Fig. 2.



height from the vertex. In overturning, its centre of gravity would describe a circular arc around the corner C. In Fig. 2, the triangle represents the section of a triangular prism, whose centre of gravity is in the point *g*, at a distance of $\frac{2}{3}$ds of its height from the vertex. The weight in the former case will act more directly to preserve the stability, while the disturbing force will act more obliquely. The former is therefore the most stable.

In the figures beneath, Figs. 1 and 2, respectively represent

Fig. 1.

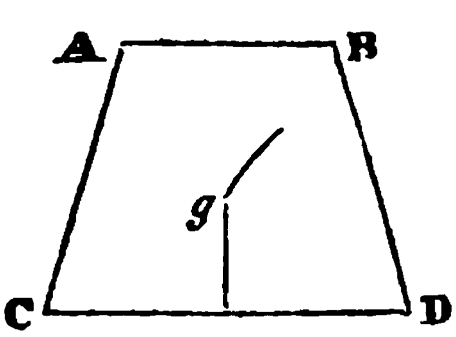
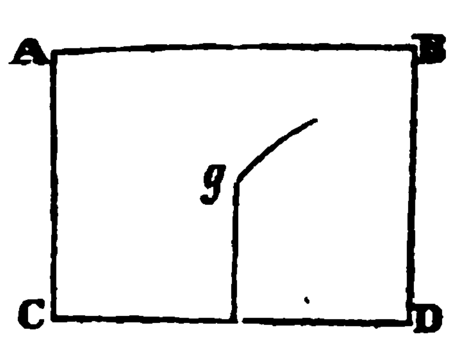


Fig 2.



sent sections of prisms, the first of which has for its section a trapezium, with parallel bases; the second is rectangular. The centre of gravity of the second is at half its height, of the first, at a distance from CD, represented by the formula

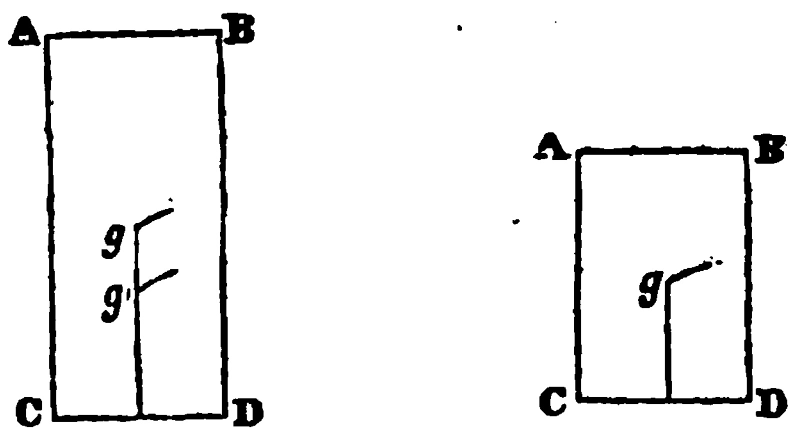
$$X=\frac{a}{3}\cdot\frac{c-2d}{c-d}$$

c being the greater and *d* the lesser base; it is obvious then that its centre of gravity will be lower than in Fig. 2, and it will in consequence be more stable.

Had Fig. 1 been the section of a truncated pyramid, the centre of gravity would have been still nearer the base, and the stability greater. Did Fig. 1 rest on the base AB, the results would be directly opposite.

In two rectangular prisms, whose sections are Fig. 1 and 2, the stability of the lowest, Fig. 2, is the greater of the two,

Fig. 1. Fig. 2.



their bases being equal; and in two prisms of equal height, but of different bases, that with the greatest base will have the greatest stability.

If however the prism, Fig. 1, should cease to be homogeneous, and be loaded with a weight towards the base CF, by means of which its centre of gravity is lowered to *g'*, whose distance from the plane of support is equal to *g* in Fig. 2, the two bodies will have equal degrees of stability.

It may hence be inferred, that pyramids and cones, of small altitude, compared with the extent of their bases, are among the most stable of all geometric figures. That walls with a broad base, and whose faces incline inwards, are more stable than those whose surfaces are parallel; that with equal bases, walls of the least heights, and with equal heights, those with the greatest bases are the most stable; that stability may be given to bodies, by constructing them in such a manner that their centre of gravity may fall below the point in which it would be if they were homogeneous.

111. A body, whose sides are inclined in such a way that it overhangs on one side of the base, may, notwithstanding, be stable, if the centre of gravity fall within the base. And even if the vertical line that passes through its centre of magnitude fall without the base, the actual centre of gravity may be so lowered, by a proper distribution of the weight, that its line of direction shall fall within the base, and stability ensue. Thus in the city of Pisa, in Italy, there is a tower that leans so much to one side,

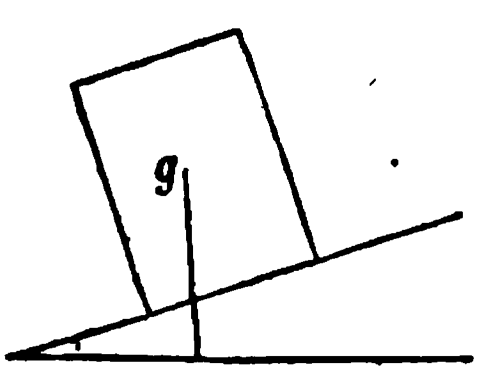
that it not only appears unsafe, but would be actually so, if homogeneous. But by an ingenious arrangement of the materials, it is rendered stable. The lower parts are built of a heavy volcanic rock; the middle of brick; and the top of a light porous stone, that will float on water: hence, the centre of gravity is so low, that its line of direction falls within the base.

112. It is not necessary that the base shall be actually a plane surface; but it is sufficient that the body rest upon points. If it rest upon no more than two points, it is in the condition of a body resting upon an edge, and the line of direction of the centre of gravity must fall in the line that joins these points, otherwise the body will not be in equilibrio. If it rest on more than two points, the base is the surface formed by joining the points by straight lines. It may in like manner rest on two edges, and the base will be defined by supposing their extremities to be joined. If the surface on which the body rests be irregular, it is best supported upon three points; for these lie always in one plane, and the state of the body is precisely the same as if this plane were applied to another. This principle is applied in practice to a variety of surveying and astronomic instruments, to which stability is given by placing them upon three feet; if they had more than three feet, they would rest firmly upon no surface but one perfectly plane, or at least having in it an equal number of points, on which to place the feet, that lie in one plane; while with three feet, they can be placed steadily on the most irregular surfaces.

A three-legged table or chair stands firmly on the most unequal floor, while one with four legs is unsteady, except upon a floor that is perfectly level.

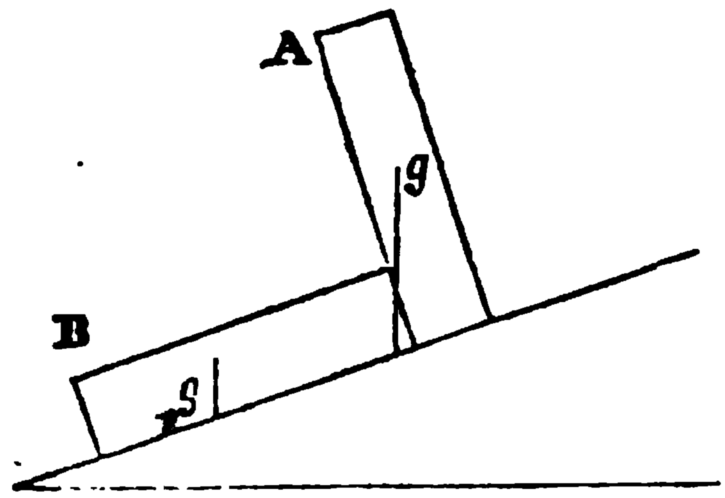
113. We have stated that there are cases in nature, in which a body will not descend an inclined plane. In such a case, if the line of direction fall within the base, the body will be stable in spite of the inclination of the plane; and if it descend upon the plane, it will slide down it.

Thus in the body beneath, if the centre of gravity be at g, its line of direction falls within the base; if the plane oppose a resistance to its descent, it is stable; but if the plane oppose no resistance, it will slide down.

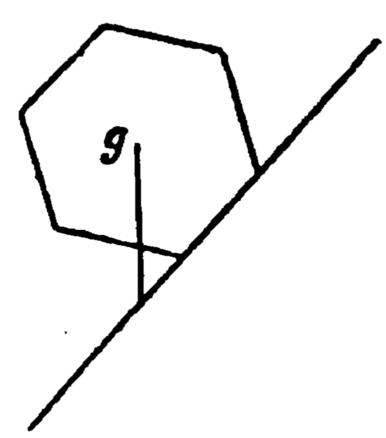


If a body be placed on an inclined plane, and the line of direction of the centre of gravity, fall without the base, it will turn around, until it apply itself to the plane by a surface, within which its line of direction will fall.

Thus the body whose section is represented beneath at A, will be overturned, and come into the position B; it will there remain at rest, if supported by a resistance in the plane, or will slide down it if not supported.

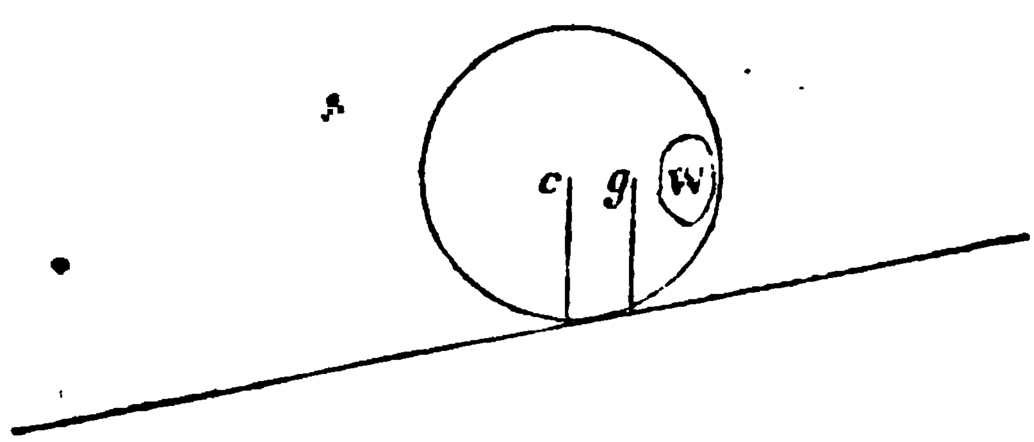


If the body have no surface within which its line of direction can fall, it will roll down the inclined plane; thus the body whose section, represented beneath, is a regular polygon, will roll down the inclined plane on which it rests.



A cylinder or sphere, having a circular section, will not rest, if homogeneous, on an inclined plane, at any of its points. But it may, if loaded by a weight which will cause the line of direction to fall upon, or above the place where it rests upon the plane, either remain at rest, or actually move up the inclined plane.

Thus in the body whose circular section is represented beneath, an eccentric weight at W will change the position of the centre of gravity from *c* to *g*, and the body will roll up the plane, until the line of direction fall upon the point at which it meets the plane, where it will come to rest.



If this weight be moved, as may be done by a spring, in such a manner as to be raised as much as it tends to fall by the rolling motion of the body, the latter will exhibit the curious phenomenon, of a body apparently mounting in opposition to gravity.

A double cone may be made to appear to roll upwards, by placing it between two edges, inclined to the horizon, and to each other, and meeting at an acute angle at their lowest points. If the double cone be laid at this angle, it rests upon its greatest section; and if the inclination of the planes to the horizon be such, that the cone, when laid at other points of the plane, shall

rest upon two of its sections whose radius has lessened more than the height of the plane has increased, the cone if laid at the place where the planes meet, will roll along them, appearing to ascend them, when in fact its centre of gravity is constantly descending.

The distinguishing property of the centre of gravity in a solid body, is, that if it be supported, the body is supported; but if it be not supported, the body will fall, and continue to fall, until it meet a resistance of such a nature as to support this point.

114. In irregular bodies, whether the irregularity arise from mere figure, or from an unequal distribution of matter throughout their bulk, the mathematical methods of finding the position of the centre of gravity are inapplicable. We may in such cases have recourse to experimental methods, whose principles are founded on the properties of the centre of gravity.

(1.) The body may be suspended alternately, from two different points in its surface. The centre of gravity will, in either case, lie immediately beneath the point of suspension; it will therefore lie at the common intersection of the two lines that join the two points of suspension to points in the body situated directly beneath each of them, when it is suspended from it.

(2.) The body may be made to rest in equilibrio, in three different positions, upon a sharp edge; the vertical plane passing through the edge, in each of the three several positions of the body, will also pass through the centre of gravity, and the common intersection of the three planes determines the situation of this point.

115. If a body move in a straight line, under the action of any other force than that of gravity, each of its particles may be supposed to be actuated by an equal and parallel force; hence it will act as if its whole mass were collected in the centre of these parallel forces. This point, which in gravitating bodies, as has just been seen, is called the centre of gravity, is, in this case, called the Centre of Inertia. Its position may therefore be found by the same processes, whether analytic, geometric, or experimental, by which the centre of gravity can be found.

CHAPTER V.

Of Friction.

116. So far as our investigations have hitherto proceeded, it might appear, that so soon as equilibrium ceases to exist, among the forces that act upon a body, it must be set in motion. This, however, does not take place in practice; for there are resistances that are incapable themselves of causing motion, and which therefore do not come within our original definition of the term force; these are yet effectual in retaining bodies at rest, after the theoretic conditions of equilibrium are at end; they are, also, capable of bringing bodies to rest, when they have been previously set in motion. Thus then, although they do not fall within the strict definition of forces, still we cannot determine the circumstances under which bodies move, without taking them into account. The retardation they produce in motions arising from other forces, is capable of being estimated in terms of a conventional unit, precisely as if they were accelerating forces, acting in directions opposed to those of the motion due to other forces. Hence we may consider the action of these resistances, precisely as if they were forces, always acting in directions opposite to those of a previously communicated motion, or to that in which a body would tend to move, when its equilibrium is disturbed. They are, in fact, passive or resisting forces, that are only called into action under certain circumstances, but which have, like active forces, a determinate measure, a definite intensity, and a known point of application.

Of such resisting or retarding forces, the more important are:

The resistance that the surfaces of solid bodies oppose to each other's motions, or that one opposes to the motion of the other. This is called Friction.

The resistance that certain bodies oppose to flexure;

The resistance of fluid media to bodies moving in them, and which solids oppose to the motion of fluids.

The two first of these are of direct importance to our present subject, and may be examined by the aid of principles that have already been laid down. The consideration of the third must necessarily be postponed, until we treat of the mechanics of fluid bodies.

117. The precise nature of friction is unknown to us, although there is a well-founded hypothesis on the subject that shall here-

after becited. We are therefore compelled to have recourse to experiment, in order to ascertain the laws its action follows. The more important of these experiments, so far as they have been recorded, are those of Coulomb, Vince, and Ximenes. To these we shall recur, describing the manner in which they were performed.

118. Friction, although always arising from the same general cause, may be classed into three distinct varieties:

(1.) That which occurs when one body slides upon the surface of another;

(2.) The friction of bodies rolling; and

(3.) The friction at the axles of wheels.

These have all been separately and fully examined, and we shall now proceed to describe the manner in which the various experiments were made, and to indicate the results that have been obtained from them.

119. If a body be placed upon a horizontal plane, and the line of direction of its centre of gravity fall within its base, it will be at rest, under two countervailing forces, its weight, and the resistance of the plane. If the plane be gradually inclined, although the equilibrium of these two forces is disturbed, because they no longer act in direct opposition to each other, the body will not at first move, but will remain at rest until the plane acquire an inclination; this inclination will be different, according to the nature of the surface and figure of the body. At this inclination the body will be set in motion, and will slide or roll down the inclined plane, according to the manner the line of direction falls. Up to the beginning of its motion, it is supported by its friction upon the plane, and this friction will be represented in direction, by a force parallel to the plane on which it moves. At the instant before motion begins, the three forces, namely, the weight, the pressure, and the friction, are exactly in equilibrio; their respective intensities may therefore be represented upon the principles in § 15.

Supposing the weight, W, to be known, and the angle i of the plane's inclination to the horizon to be determined, the value of the friction, F, and pressure, P, will be

$$F = W \sin. i, \quad P = W \cos. i;$$

and the friction will be given in terms of the pressure, by the expression

$$F = P \tan. i.$$

Experiments conducted upon this principle, have given the following results:

(1.) The friction is greatest between rough surfaces, and diminishes with the degree of polish that is given to them.

(2.) Friction is greater, all other things being equal, between the surfaces of bodies that are of the same material, or homogeneous, than it is between bodies of different materials.

(3.) The rubbing surfaces remaining the same, the friction is directly proportioned to the pressure.

(4.) The friction does not increase or diminish with the area of the rubbing surface, the weight and the nature of the surface remaining the same.

These experiments are limited, by their very nature, to the determination of the resistance that prevents a body from being set in motion; and it might, at first sight, appear more than probable, that this force is more intense than the friction which retards the velocity of a moving body. Neither do they give any information whether the intensity of friction bears any relation to the velocity. Another defect arises from the small number of the experiments that have been performed in this manner, and we are hence uncertain, whether the results are applicable to all cases whatsoever, or limited to a few particular instances.

120. The experiments of Coulomb, Vince, and Ximenes, were performed in another manner. A body was drawn along a horizontal table, by means of a weight attached to it by a cord, and this cord passed over a pulley. The weight that produces a constant velocity is obviously the measure of the friction, which is, in this case, the resistance that opposes the *motion* of bodies; the weight necessary to set them in motion, is the measure of the resistance that opposes their passage from a state of rest.

The different circumstances to be examined are:

(1.) The relation of the friction to the pressure;

(2.) The effect of the nature of the surfaces, and of the manner of their preparation, upon the friction;

(3.) The variation in the friction produced by a longer or shorter continuance of the contact, previous to the application of a force to overcome it;

(4.) The influence of the extent of the surface;

(5.) The change, if any, at different velocities.

(1.) As respects the relation of the friction to the pressure.

When at a maximum, all other circumstances remaining the same, the friction was found to have a constant relation to the pressure. This maximum, as will hereafter be seen, is that which opposes the force that acts to set a body in motion.

The friction of surfaces, that had been at rest until the maximum was attained, was found to be as follows, taking the mean of the experiments:

Of Oak resting upon Oak, the fibres being parallel,	$\frac{1}{8}$
Of Oak resting upon Oak, the fibres being at right angles to each other,	$\frac{1}{3.75}$
Of Oak resting upon Fir,	$\frac{1}{1.5}$
Of Fir resting upon Fir,	$\frac{1}{1.78}$
Of Iron resting upon Oak,	$\frac{1}{5.5}$
Of Iron resting upon Iron,	$\frac{1}{3.5}$
Of Iron resting upon Brass,	$\frac{1}{3.8}$
and if well polished, and the surfaces small,	$\frac{1}{6}$

The friction of bodies in motion was found considerably less, being at a mean as follows:

Of Oak moving on Oak,	$\frac{1}{9.5}$
Oak moving on Fir,	$\frac{1}{6.3}$
Fir moving on Fir,	$\frac{1}{6}$
Elm moving on Elm,	$\frac{1}{10}$
Iron or Copper moving on wood,	$\frac{1}{13}$
Iron on Iron,	$\frac{1}{3.55}$
Iron on Copper, after long attrition,	$\frac{1}{6}$

(2.) It was found that the metals could be polished separately, until the minimum of friction was attained, but that in soft substances, it was necessary that they should be compressed, either in the course of the experiments, or by previous pressure. Thus in newly planed wood, the maximum friction was not far from equal to the pressure, while after being some time in use, it fell to the ratio at which it has been stated, of $\frac{1}{3}$.

The polishing of the surfaces of iron and copper, rubbing against each other, reduced the friction nearly one half.

Unctuous substances, interposed between the rubbing surfaces, diminish the friction; and the experiments of Coulomb showed, that those which are hardest have the greatest effect, when the

weight is great; thus, although in light and delicate machinery, the most limpid oils are best, they are wholly ineffectual in great pressures, when tallow must be employed. At still higher pressures than those which became the subject of experiment, it appears from practical observations, that tallow loses its power of diminishing friction. So also is it less effective when the velocity becomes sufficient to melt it. In this case it acts as if it were oil, being well suited to diminish friction when the weight is light, but of little effect in great pressures. When oleaginous substances cease to have effect, as tallow, when the pressure becomes too intense, or the heat produced by the velocity is efficient to render it fluid, solids of an unctuous texture have been found to answer the purpose. Thus in carriages moving rapidly, and in large machinery, plumbago has been mixed with tallow: it has also been used separately; and recently, soap-stone (steatite) has been found efficacious in lessening friction when the weight was so great that all other means failed.

Within the limits of the experiments, the mean friction, when oleaginous matters were interposed, was found to be as follows:

Tallow interposed between two surfaces of oak, - - - - -	$\frac{1}{28}$
If wiped off after application, - - - - - - - - - - - - -	$\frac{1}{14}$
In very small surfaces, - - - - - - - - - - - - - -	$\frac{1}{17}$
Tallow interposed between wood and oak, moving slowly on each other, - - - - - - - - - - - - - - - -	$\frac{1}{35}$
Between brass and oak, under similar circumstances, - - -	$\frac{1}{47}$
After being sometime in use the friction was increased to thrice these amounts, or to - - - - - - - - -	$\frac{1}{12}$ and $\frac{1}{16}$.

The same result took place at increased velocities.

In the case of tallow, interposed between metallic surfaces, the frictions were the same at all velocities.

Tallow, interposed between two surfaces of iron, gave - - -	$\frac{1}{10}$
Between iron and copper, - - - - - - - - - - - - -	$\frac{1}{11}$

While

Oil interposed between the same surfaces, gave for the friction,	$\frac{1}{8}$

(3.) It was found, that in the case of the metals resting upon each other, the maximum of friction was reached instantly; in wood resting upon wood, the maximum was reached in a few minutes; in the contact of metals with wood, the maximum was

not attained until some days had elapsed; and when grease was interposed between any substances whatsoever, the time during which the friction continued to increase, was still longer than in the latter case.

The relations that have been stated in the two first instances, were not found to be wholly independent of the amount of pressure. That is to say, the ratio of the surface to the pressure was not constant, but appeared to diminish at the higher pressures, that were the objects of experiment. This diminution was even more obvious, in the experiments of Vince and Ximenes, than in those of Coulomb.

(4.) The magnitude of the surface was found to have an appreciable effect upon the friction. In the case of oak moving upon oak, in pressures from 100 to 4000 lbs. per square foot, an adhesion or additional resistance was found amounting to about $1\frac{3}{4}$ lbs. per square foot; and in all other cases, an analogous increase was remarked.

(5.) Under similar circumstances, the weight, that overcame the friction, was found to be nearly the same at all velocities. The slight variations that were observed, seemed rather to lead to the conclusion, that at small velocities the friction increases with the velocity, but that at great velocities, it diminishes in the ratio of some small function of the velocity.

It thus appears that the variation in pressure, in magnitude of surface, and in velocity, have but little effect, unless they become very great.

We may therefore, in all usual cases, assume the following laws as sufficiently accurate for practical purposes.

(1.) Between similar substances, under similar circumstances, Friction is a constant retarding force.

(2.) Friction is greatest between bodies whose surfaces are rough, and is lessened by polishing them.

(3.) It is greater between surfaces composed of the same material, than between bodies composed of different materials.

(4.) If the rubbing surfaces remain the same, the friction increases directly as the pressure.

(5.) If the pressure continue the same, the friction has no relation to the magnitude of the surface.

(6.) The relation between the pressure and the friction, cannot be safely taken at less than $\frac{1}{3}$, in calculating the effect of prime movers.

On the other hand, when friction is to be substituted for a fixed resistance, it cannot safely be estimated at more than $\frac{1}{37}$.

Although the weight, which measures the intensity of friction, be constant, whatever be the velocity, still as the measure of a force is not merely the weight that it is capable of raising, but de-

pends also upon the rate at which the weight is lifted, the power that is applied to overcome the constant friction, must increase with the velocity at which that resistance is overcome.

The several laws that have just been laid down, are, as is obvious from the result of the experiments, not absolutely, but only nearly true, and cases occur occasionally in practice, in which the minute effects that are due to the increase of pressure of the surface, and of the velocity, become important. Thus: in launching a ship, the vast weight which it has, appears to become a force capable of very much lessening the friction, and the vessel descends down a plane of less inclination than a lighter body would. At the Shoot of Alpnach, in Switzerland, where trees of great size are conveyed along a trough of but small inclination, for 7 or 8 miles, in consequence of a velocity previously acquired in falling through a curved spout of great inclination, the phenomena, as noted by Playfair, are such as can only be explained by assuming, that the friction of great masses, moving with great velocities, is considerably less than that of smaller bodies moving slowly.

121. The friction of rolling bodies has also been investigated experimentally by Coulomb. The following are deductions from his experiments :-

(1.) Like the friction of sliding bodies, it is a constant force;

(2.) It is affected by the nature of the surface, so far as polish is concerned, but is not lessened by the interposition of oleaginous and unctuous substances;

(3.) It is less between heterogeneous than between homogeneous substances;

(4.) It is directly proportioned to the pressure;

(5.) It has no relation to the magnitude of the surface;

(6.) Its measure is much less than in the case of sliding surfaces, and varies in the inverse ratio of the diameter of the rolling body.

A cylinder of lignumvitæ rolling on rulers of different kinds of wood, and having a diameter of 32 inches, was not resisted by a friction, at a mean, of more than $\frac{1}{123}$.

122. The friction of the axles of wheels is still of another description; it is less than that of sliding, and more than that of rolling bodies. It follows, in all respects, the general laws of sliding bodies. An axle of iron, turning in a box of oak, had a friction of $\frac{1}{7}$; when both were of wood, the friction was $\frac{1}{12}$.

In metallic axles, resting in boxes of another metal, and well coated with grease, the friction has been found to be no more than $\frac{1}{20}$.

Of wooden axles in wooden boxes, when coated in a similar manner, from $\frac{1}{16}$ to $\frac{1}{33}$.

Of iron in wood, also coated in grease, $\frac{1}{20}$.

123. To adopt the theory of Coulomb, friction appears to arise from the porosity of bodies. All, even the most dense, have large spaces between the particles of which they are composed. When they rest upon each other, the prominent parts of the one fall into the cavities of the other, and are in a manner locked. In hard substances, the maximum of this effect will be produced in a short time; but where they are soft, the maximum will not be reached until the utmost compression the pressure is capable of producing, is attained.

The polishing of bodies consists merely in rubbing down the asperities, and multiplying the cavities in number, but diminishing their depth; hence they still interlock; but the weight must be raised but a small distance, in order to disengage the prominences from the cavities, and the more perfect the polish the less will be the height to which the weight must be lifted. The friction of bodies in motion must obviously be less than that of bodies at rest, because the projections of one substance require a certain definite time to adapt themselves to the cavities of the other, and the difference will be greatest in those bodies that require the greatest time to adapt themselves to each other. When unctuous matters are interposed, they fill up the cavities and prevent the penetration of the prominent parts; hence the resistance becomes almost solely that which is due to the attraction of their own particles. As the weight increases, liquids being more readily forced out of vessels that contain them, and thus most easily disengaged from the cavities, oppose less resistance than solids to penetration, and are less efficacious in diminishing friction; but on the other hand, the more perfect the fluidity, the more easily are the particles of liquid moved among each other; and hence, so long as the pressure is not sufficient to force them out of the cavities, they will be better suited to diminish friction than thicker oils, or solid grease.

The particles of homogeneous bodies are arranged in a similar manner, whether by the action of crystallization, or their organization; hence the cavities and asperities will fit better, and apply themselves more closely, than in heterogeneous bodies.

When a body slides upon another, one of two things must occur, either the weight must be partially lifted, or the asperities must be broken down before the sliding body; either of these will require a force of some intensity, but when a body rolls, the very act of rolling disengages the prominences from the cavities. The action of axles, resting in sockets, is intermediate between that of

rolling and that of sliding bodies, and hence has an intermediate degree of advantage. It will be seen hereafter, that an adhesion takes place between bodies which depends upon the extent of the surfaces, and although this be extremely small, it is sufficient to account for the small increase that follows the law of the surface.

The friction being in a certain degree removed from the bodies themselves, and taking place among the particles of the oleaginous substances, when the latter are used as coatings; and as an increased pressure will have an effect in overcoming the latter resistance, it appears probable, that in this case, an increased pressure may act in opposition to the other part of the friction, of which it is the effective cause.

The property that bodies have of moving forward in the straight line in which the force applied has been directed, would prevent the prominent parts of bodies, moving with greater velocities, from entering as deep into the cavities of those on which they move, as they would if moving with less velocities, and hence, at great velocities, there ought to be an ascertainable diminution of the friction.

124. The whole of the circumstances that are involved in friction, may be represented by a formula, which is as follows:

$$F = fP - \varphi P + \varphi' S - \varphi'' V. \qquad (108)$$

In this, F represents the friction, P the pressure, S the area of the surface, and V the velocity; f is the ratio of the pressure to the friction, under ordinary circumstances, say the fraction given for the particular cases in §120; φ is a very small function of the pressure, whose amount has not been fully established from experiment; φ' is the cohesion of the unit of the surface; and φ'' a very small coefficient of the velocity, whose magnitude is also unknown.

125. The above theory, and the inferences from the experiments whose results have been cited, point out the modes that may be employed in practice to lessen or overcome the friction.

(1.) The line in which the power is applied, instead of being parallel to the surface on which the body moves, may be slightly inclined upwards. If the power be resolved into two components, one of which is parallel, and the other perpendicular to the surface, the first alone will be applied to the draught, and the other will act to raise the weight of the body. By this latter action, the parts of the surfaces that have been interlocked may be disengaged.

(2.) The highest practicable degree of polish must be given; and the surfaces except in the case of rolling, coated with unctuous matters. When the pressure is small, the most limpid oils should be used; as the pressure increases, those of more viscidity; on a

still further increase, tallow must be employed; when the pressure becomes very great, or the velocity is such as to melt the tallow, plumbago may be mixed with that substance; and finally, in the greatest pressures that are to be found in practical mechanics, plumbago alone, or soapstone, both in the state of fine powder, may be employed.

(3.) The motion of rolling may be substituted for that of sliding, when the body has a figure that will admit of it.

Thus tobacco in Virginia was formerly drawn to market, by making the hogshead roll along the ground. This change of the mode of motion, is not only advantageous in bodies of a circular section, but may be made useful in others, although in them it will become necessary to lift the weight at each turn the body makes. If the force necessary to lift the weight, in such cases, be not greater than the difference between rolling and sliding frictions, an advantage will be gained.

(4.) If the body be of such a figure that it cannot advantageously be made to roll, it may be set upon rollers, and the advantage which is due to their diameters and mode of motion, will be attained. The body, in moving forward, will leave the rollers behind it, which must therefore be lifted and carried forward to receive it again; hence this method is, generally speaking, confined to short distances. When, however, the weight is very great, it may be more advantageous than any other practicable means: thus, for instance, the great rock, that forms the pedestal of the statue of Peter the Great, at St. Petersburgh, was moved a distance of several versts, from the place where it was found imbedded, to the banks of the Neva, upon small spheres of metal, laid in a trough; and after it was transported to the city, was carried by the same method to the place where it was to be set up. The removal of this vast mass would have been impracticable, by any of the ordinary means of transportation.

(5.) When rollers are inapplicable, the weight to be moved may be laid upon a wheel carriage: here, besides the diminution of friction which is obtained, a mechanical advantage is gained, equivalent to the ratio of the diameter of the wheel to that of its axle.

(6.) The use of wheels being attended with this mechanical advantage, the extremities of their axles may be made to rest upon the circumferences of other wheels, by means of which a similar advantage may be gained, or may be made to press against revolving bodies, instead of resting in cylindrical sockets. Such applications of this principle are called friction wheels, and friction rollers.

The most beautiful application of this kind, is that which was adapted by Atwood to his machine, which has already been spoken of in § 95. In this apparatus, the extremities of the axles of the wheel, over which the cord, that connects the weights, passes, are made each to rest upon the circumference of two other wheels, and a mechanical advantage is gained in the ratio that has just been mentioned. By this arrangement, the friction is rendered wholly insensible, and interferes in no appreciable degree with the results of the experiment.

In the patent blocks of Garnett, the axles, instead of moving in cylindrical sockets, rest each upon six friction rollers, arranged in a box, and by this means a similar advantage is gained.

An attempt, founded upon similar principles, is now making by Wynans of New-Jersey, and applied to the wheels of carriages.

It will be obvious, that the advantage of wheels ceases, when their effective friction becomes greater than that of their circumferences upon the surfaces upon which they move. This is the case upon hard smooth substances, such as ice, in which case sledges are to be preferred to wheel carriages.

(7.) A knowledge of the fact, that a very great diminution of the rubbing surface is attended with a sensible diminution of the friction, may be applied with advantage in a limited number of cases. Thus: where the surfaces are so hard as to admit of no penetration, even where progressive motions are employed, the moving body may have its surface diminished almost to an edge. Of an application of this sort, we have an instance in the case of skates. In rotary motions, the application of this principle is more easy. Thus: although in machinery, generally, the axles must be of a certain size, in order to bear the weight of the wheels, and often the action of other pressures to which they are subjected, these weights and pressures become so small in the case of part of the train of wheels in the common watch, that the bearing of the axles may be reduced to the smallest points that can be made on hardened steel. These points, instead of resting in sockets, are supported in small shallow cups of a hard material, agate, ruby, or diamond. In motions of oscillation upon axes, the axis may take the form of an edge of steel, and may be made to rest upon a cylindrical surface, or even upon a polished plane of some hard material; of this we shall have instances in the Balance and the Pendulum.

126. Friction is by far the most influential of the causes, by which bodies moving near the surface of the earth are brought to rest. If supported, they experience a friction from the body that supports them; if unsupported, they fall to the earth, either

in a vertical or inclined direction; if in a vertical direction, the friction they meet in penetrating, rapidly destroys their motion, even if the earth be soft where they fall: if in an inclined direction, of the two components of their motion, one of which is perpendicular, the other parallel to the surface of the earth, the former is at once destroyed by the resistance to penetration, the other remains to carry the body along the surface, and this again would be finally destroyed, by friction against the surface, even did no other retarding force act.

CHAPTER V.

OF THE STIFFNESS OF ROPES.

127. It frequently becomes necessary, in practical mechanics, to bend ropes over cylinders and rollers. This is the case even in some of the elementary machines. When ropes are thus bent, they always oppose a resistance, to overcome which it becomes necessary to apply a part of the force that actuates the machine. This resistance, it is obvious, may vary:

(1.) With the tension of the rope, or the weight by which it is stretched;

(2.) With the quality of the rope, depending upon the nature of its materials, and the manner of its manufacture;

(3.) With the size of the rope;

(4.) With the diameter of the cylinder, over which it is bent.

128. The best experiments on this subject are also by Coulomb. The first of the results obtained by him is: that like friction, the resistance of ropes to forces applied to bend them, is a constant retarding force. The deductions, in respect to the circumstances that have been stated, are as follows, viz.:

(1.) The resistances of ropes are directly as the tensions to which they are subjected;

(2.) The resistance is greatest in ropes that have been strongly twisted, in ropes coated with tar, and in new ropes. The ratio of these is in some measure included in the next circumstance.

(3.) The resistance increases with some determinate power of the diameter of the rope, which we shall call n:

In new tarred ropes, $n=2$,
In new white ropes, $n=1.7$,
In old ropes, $n=1.5$.

(4.) The resistances are inversely as the diameters of the cylinders, around which the ropes are bent.

129. When a rope is wound more than once around a cylinder, it is found that the resistances increase in geometric progression.

This principle is frequently applied in practice, when it is wished to oppose rapidly increasing resistances to moving bodies: thus, in arresting the progress of a vessel, a rope is turned again and again around a post; and a small number of turns will become efficient to overcome any force not of sufficient intensity to break the rope.

The resistance, in any particular case, is represented by the formula,

$$m + p\,w, \tag{109}$$

when m is the absolute resistance of the rope, and p the proportion of the weight w, that is necessary to be added, in order to overcome the increased resistance due to the addition of a weight.

130. To furnish data for the application, it is sufficient to quote a single instance calculated from the experiments of Coulomb, whence, by the principles we have laid down, the resistance of any other rope may be calculated. A rope not tarred, of an inch in diameter, may be bent around a cylinder of 4 inches in diameter, by a weight of $\frac{1}{10}$th of a pound, added to $\frac{10}{580}$ of the weight by which it is strained; a tarred rope, of the same size, requires a weight of about $\frac{1}{5}$th of a pound, added to $\frac{1}{50}$th of the weight by which it is strained. From these two instances, the resistance of any other rope may be calculated by means of the principles that have been laid down.

The sizes of ropes are usually estimated by the measures of their circumference, as 1, 2, 3 inch, &c. Hence the instances given may serve as units.

The resistance, in all cases whatsoever, will be

$$\frac{mc^n + pwc^n}{\frac{1}{4}d}. \tag{110}$$

In which expression, p, w, and m are as in the former equation, and have the values given in our instance; n is the power from § 128; c the circumference of the rope; and d the diameter of the cylinder, both expressed in inches.

CHAPTER VI.

Of the Mechanic Powers.

131. A machine is an instrument, by means of which we change either the direction or the intensity of a force, or both its direction and intensity. The general principle of the equilibrium of all machines whatsoever, is to be found in that of virtual velocities, By this, if the several points of the machine, on which the forces act, were each to be supposed to move under the action of the force that is applied, equilibrium will exist when the sum of the products of all the forces into their several velocities, the latter being distinguished as positive, or negative, according to their directions, is equal to 0. As machines, generally speaking, have a fixed point, the proposition, in conformity with § 71, case 5th, becomes: Equilibrium will exist in any machine, when the sum of the products of all the forces, on each side of the fixed point, into the respective virtual velocities of their points of application, is exactly equal to the sum of the similar products on the opposite side of the fixed point. The value of this principle, as applied to the useful properties of machines, will be discussed hereafter. For the present, we shall leave it, and proceed by more direct methods to investigate the conditions of equilibrium in the more simple forms of machines.

132. Machines are either simple or compound. The former are the elementary parts, of which all compound machines are made up, by combinations of various descriptions; they are also capable of being used singly. These simple machines are called the Mechanic Powers.

133. The Mechanic Powers are six in number, viz.: The Lever, the Wheel and Axle, the Pulley, the Wedge, the Inclined Plane, and the Screw.

It has already been stated, that their properties may be reduced to a single principle; but in the mode that has been chosen for examining their conditions of equilibrium, it will be seen, that the properties of four of them may be included in those of two others. Hence the Mechanic Powers are arranged in two divisions: to the first belong the Lever, the Wheel and Axle, and the Pulley; to the second, the Wedge, the Inclined Plane, and the Screw.

Of the Lever.

134. A lever is an inflexible bar, or rod, resting upon a fixed axis or prop, that is called the Fulcrum, around which it is free to move, under the action of the impressed forces. The general condition of equilibrium, in the case of any number of forces whatsoever, is an immediate deduction from the theory of the moments of rotation in § 34, and is as follows:

135. In any lever, whatsoever, equilibrium will exist, when the sum of the products of the forces applied to it, on one side of the fulcrum, into the perpendiculars let fall from that point upon their respective directions, is exactly equal to the sum of similar products, on the opposite side of the fulcrum.

Among the forces, it is obvious, that the weight of the lever itself may be included, its direction being a vertical line, and its point of application the centre of gravity of the bar.

136. The properties of the lever, and indeed of all the simple mechanic powers, are most usually limited to the case of the action of no more than two forces. One of these is called the Power, the other, the Weight. By the Weight, we understand the resistance to be overcome; by the Power, the force, whatever be its nature, that is applied to overcome the resistance. In the usual mode of treating the theory, the bar is supposed to be devoid of weight. The points of application of these two forces, and the fixed point, or fulcrum, may have three possible positions in respect to each other, and we hence distinguish three different kinds of lever.

(1.) When the fulcrum is between the power and the weight.
(2.) When the weight is between the power and the fulcrum.
(3.) When the power is between the weight and the fulcrum.

137. If the lever be straight, and the power and weight act parallel to each other, equilibrium will exist, when the power is to the weight in the inverse ratio of their respective distances from the fulcrum;

or when $$Pa=Wb. \qquad (111)$$

This case becomes the simple one in § 22, of finding the point of application of the resultant of two parallel forces; for the fixed point will be in the same state, as if it were acted upon by a force equal and opposite to the resultant of the other two; and it was there demonstrated, that the point of application of the resultant of two parallel forces, divides the line of application into parts inversely proportioned to the intensities of the two forces.

138. In the case of a bent lever, or when upon a lever, whether straight or crooked, the directions of the power and weight are not parallel, equilibrium will exist when the two forces are to

each other inversely as the perpendiculars let fall from the fulcrum upon their respective directions.

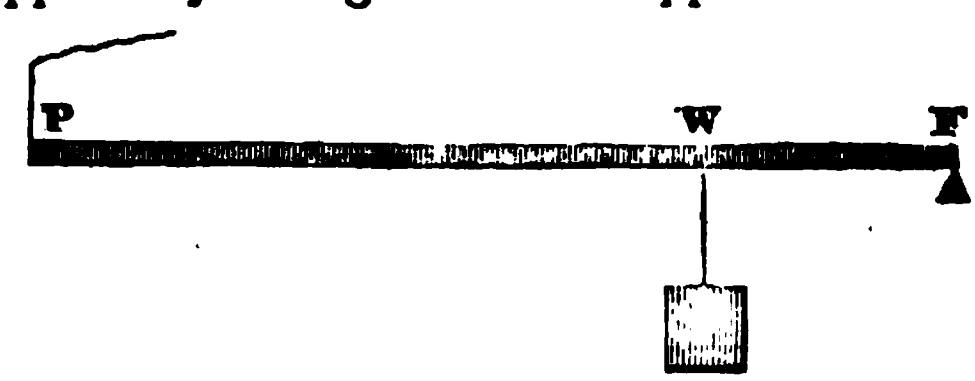
This may be deduced directly from (25), it being again obvious, as in the former case, that the fulcrum must be the point of the application of the resultant of the two forces that keep the lever in equilibrio; and by that equation it is shown, that this point is situated in such a manner, that the perpendiculars let fall from it, upon the directions of the two forces, are in the inverse ratio of the intensities of the two forces.

139. The lever is perhaps the most ancient, as it is still the most familiar in its use of all the mechanic powers. The instances of its practical application are almost too numerous to admit of their being named. It will be sufficient to give a few, merely as illustrations of its properties.

(1.) Of the first kind of lever, where, as in the figure beneath, the fulcrum is between the power and the weight, we have in-

stances in the common crow-bar and handspike, in scissors, pokers, pincers, snuffers; and of a bent lever of this description, in a hammer, when used for drawing nails.

(2.) In oars, the fulcrum is where the blade strikes the water; the weight is applied where the oar rests against the side, and the power is applied by taking hold of the opposite extremity; hence



they are levers of the second kind, in which, as in the figure, the power is between the fulcrum and the weight.

The rudders of ships act upon similar principles. Of the same kind of lever, cutting knives fixed at one end, doors moving upon their hinges, the manner in which a weight is borne upon a wheel-barrow, nut-crackers, &c., may be cited as instances. In these two first kinds of lever, the distance of the power from the fulcrum is greater than that of the weight; hence the weight has a greater intensity than the power, when the two are in equilibrio, and thus the power is capable of overcoming a resistance greater than its own measure.

(3.) Of the third kind of lever, in which, as represented on the figure, the power is between the weight and the fulcrum, we

have examples in the common tongs, in sheep shears, in the manner a ladder is raised against a wall. In this kind of lever, the power being nearer to the fulcrum than the weight, the former must have, in the case of equilibrium, a greater intensity than the latter. In the two first kinds of lever, then, a given power will raise a greater weight, or overcome a greater resistance than it can when it acts directly; while, in the third kind of lever, the weight raised, or the resistance overcome, is less than the power is capable of doing, if it act without the intervention of the lever. If the several figures be referred to, it will be seen that these advantages in the two first cases, and this disadvantage in the last, are compensated by the difference in the velocities of the points of application, in case motion should take place. These velocities will be represented by the arcs described. These are to each other as the radii of the respective circles, of which they are similar arcs, and as their radii are the arms of the levers, that are to each other in the inverse ratio of the forces applied at their extremities, the product of the weight into its velocity, will be equal to that of the power into its velocity. This, it will be at once seen, is no more than a case of the general principle of virtual velocities. Hence, whenever intensity of force is gained by means of the lever, it is always at the expense of an equal loss of velocity; and where velocity is gained, it is gained at the expense of power. A similar inquiry into the velocities, with which the points of application of the power and weight would move, in the case of the equilibrium being disturbed, would show that the same relation exists between the intensity of force, and the velocities lost or gained in actual motions.

140. The Balance is one of the most useful applications of the lever. It is no more than a lever of the first kind, with arms of equal lengths, resting upon a fulcrum. To the two extremities are attached pans, or scales, in which heavy bodies may be placed. It will be obvious, that when the two weights are equal, the balance will be in equilibrio under their joint action. If then a certain number of units and fractions of any conventional system

of standard weights, be in one of the scales, it will just counterpoise a substance placed in the other, whose weight must have an equal value in that system. Hence, by means of a balance, the unknown weight of any articles whatever, may be determined by the aid of a set of properly graduated weights.

The balance, having this property, is of the most extensive utility, not only in philosophical inquiries, but also in every variety of trade, in which the articles cannot have their quantities determined by measures of length, of surface, or of capacity, either on account of their absolute nature, or in compliance with custom.

141. It is difficult, in practice, to make the distances between the two points whence the scales are suspended, and the fulcrum, exactly equal. Still a sufficient degree of accuracy may be attained for all the purposes of trade. But it sometimes happens that balances, either by accident or design, have arms of unequal lengths. In this case, the error may be detected by changing the position of the two counterpoising weights from one scale to the other. If, when thus transferred, they are still in equilibrio, the balance is true, if they are not, it is false.

The actual weight may be obtained by means of a false balance, by weighing the substance whose quantity is to be determined, successively, in the two scales. Its true weight will be the geometric mean between the known weights that counterpoise it, in the two different positions.

Let P be the absolute weight of the substance whose quantity is to be determined; W and W′, the known weights that counterbalance it in the two different scales; a and b the two arms; then from the property of the lever,

$$Pa=Wb, \tag{111}$$
$$Pb=W'a;$$

whence

$$P^2=WW',$$

and

$$W : P : W'. \tag{112}$$

When a balance is used for delicate investigations, any error that might arise from an inequality in the arms, is readily obviated by a simple process. The body, whose weight is to be determined, is placed in one of the scales, and is counterpoised by a substance capable of minute division, such as fine sand, placed in the other. When the balance has been brought to rest in a truly horizontal position, the body to be weighed is removed, and weights placed in the same scale, until they counterpoise the substance that remains in the other scale. It will be obvious, that the body whose weight is required, and the standard weights

being both in equilibrio with the same substance, and having acted upon the same arm of the lever, must be exactly equal to each other.

142. The properties of a good balance are—

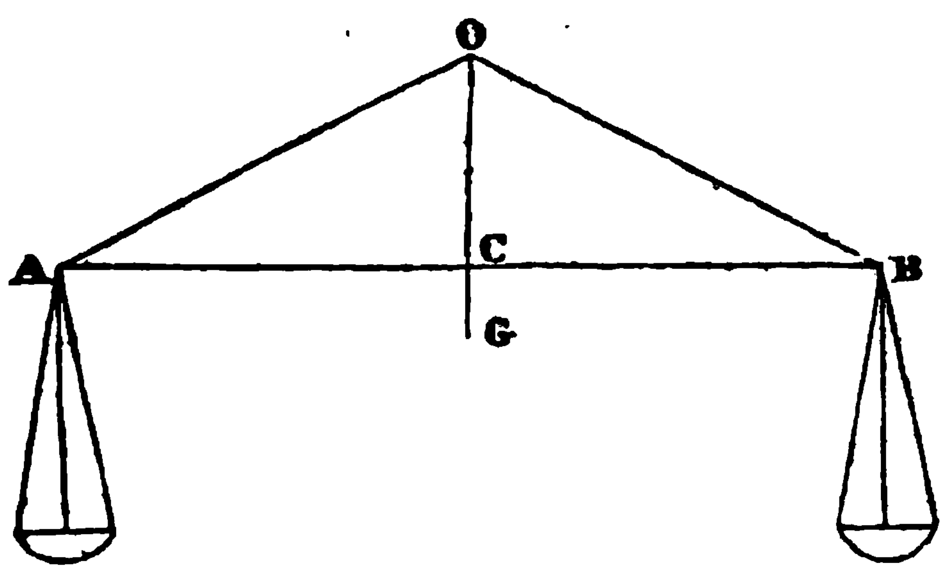
(1.) That it should rest in a horizontal position when loaded with equal weights, and in an inclined position when the weights are not equal;

(2.) That it should have great sensibility, so that a small proportion of the weight with which it is loaded, added to either scale, shall disturb the equilibrium;

(3.) That it should be stable, or soon return to rest, after having been put in motion by a change of the weights.

These properties depend in part upon a proper choice of the point of suspension, in respect to the positions of the line that joins the points whence the scales are suspended, and of the centre of gravity; and in part upon accurate mechanical construction.

In the figure beneath, let A and B be the points whence the



scales are suspended; G, the centre of gravity; O, the centre of suspension; C, the point where the lines that join A,B, and O,G, intersect each other.

If the points O,C,G, were to correspond, the balance would be, § 107, in a state of indifference, it would be the most sensible to variations of weight, but would have no tendency to come to rest in a horizontal position. The higher the point O, the more stable will be the equilibrium, but the less, all things else being equal, will be the sensibility. The longer the arms, the greater will be the moments of rotation of the weights, and consequently the greater the sensibility. The centre of gravity will be lowered by additional weights in the scales, and in this way also the sensibility will be diminished.

If the point O should fall below C, the balance will be unsteady;

but if below both C and G, the balance will be in a state of tottering equilibrium, and will be liable to be overturned.

In order that the balance shall come to rest in a horizontal position, not only must its arms be of equal lengths, but they, with the scales attached, must be of equal weights.

In the actual construction of the balance it is important—

(1.) That the motion shall be attended with as little friction as possible: this is effected by making the axis of suspension of the form that is called a knife-edge. A prismatic bar of hard steel is passed through the beam of the balance, and is formed into an edge beneath, by the intersection of two convex curves. In the common balance, this is made to rest at each end on the surface of a hollow ring or cylinder.

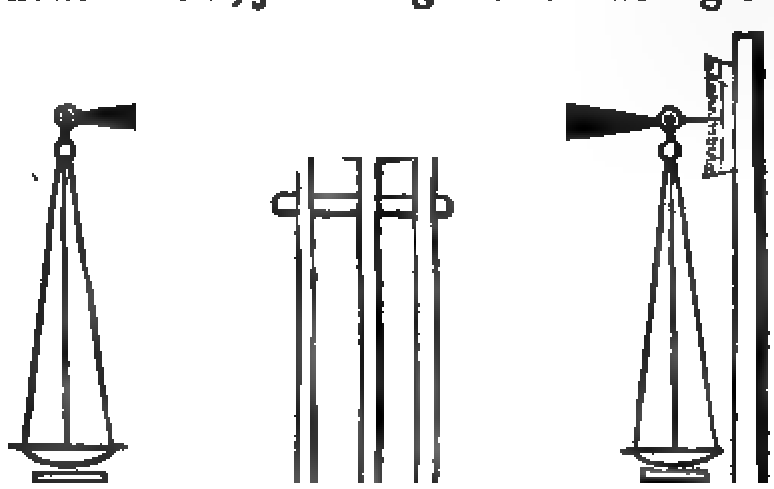
In some of the more accurate balances, the knife-edges rest upon planes of polished agate. This is one of the cases that were quoted in § 125, in which the friction is reduced to the minimum, by diminishing the rubbing surface to a mere edge.

As the friction is in the balance, as in other practical cases, proportioned to the pressure, the greater the weight with which the balance is loaded, the less will be the sensibility; and the latter is, as has just been shown, also diminished, by the lowering of the centre of gravity. To obviate this defect, some balances have been made with a sliding weight beneath the point of suspension, by changing the position of which, the balance may be made more or less sensible.

(2.) The distance between the centre of suspension, and the points whence the scales hang, ought to remain exactly the same during all the oscillations of the balance. This is sometimes effected by suspending the scales from knife-edges also. These are passed through the ends of the beam with the edges uppermost; and the scales are hung from hooks or rings that rest upon them. Sometimes, to make the touching surfaces the least possible, these rings are ground on the inside to a sharp edge.

A balance of a good construction is represented beneath.

In some of the best balances by Ramsden and Troughton, the beam, instead of being a bar, is made of the form of two similar and equal hollow cones, joined together at their greater bases.



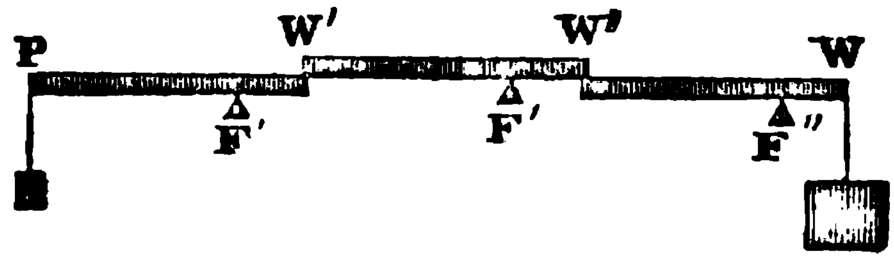
Such a figure possesses far more strength than a solid bar. It may, therefore, be made much lighter than almost any other form. It is said that these instruments weigh so well, as to note

differences of one-millionth part of the weight with which the scales are loaded.

It is, however, an excellent balance that will weigh to the $\frac{1}{100000}$th of the weight with which it is loaded, and generally speaking, balances do not weigh more nearly than from $\frac{1}{25000}$ to $\frac{1}{10000}$th of the weight.

Balances of different sizes, strength, and materials, have different degrees of accuracy. For weighing weights of different magnitudes, then, several balances will be necessary, from those which turn with weights of a small fraction of a grain, to those which will bear several tons.

143. Levers may be combined together in such a manner as to



form a compound machine, as is the case in the figure.

Here the power P would be in equilibrio with a force acting at W′, when their relation was inversely as their distances, or

$$Pp = W'w;$$

but, if instead of a weight, the end W′ be made to act upon the extremity of another lever, whose arms are p' and w', then

$$Wp' = W''w';$$

and the third lever will have the following condition of equilibrium:

$$W''p'' = Ww'';$$

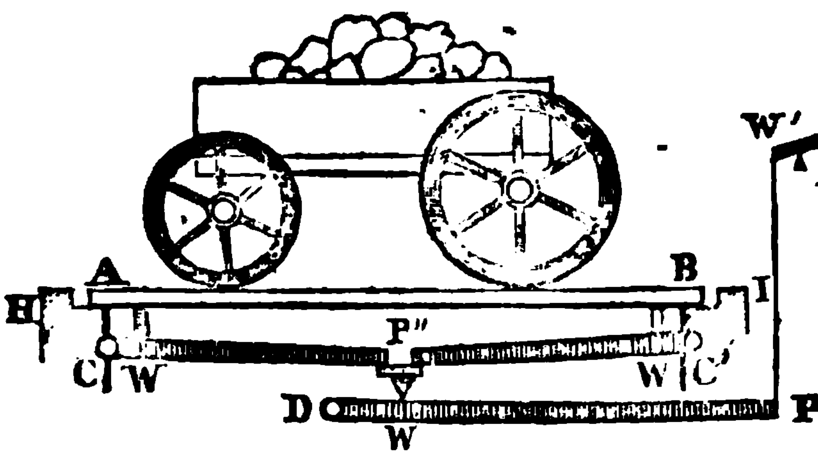
whence

$$Ppp'p'' = Www'w' : \qquad (113)$$

therefore,

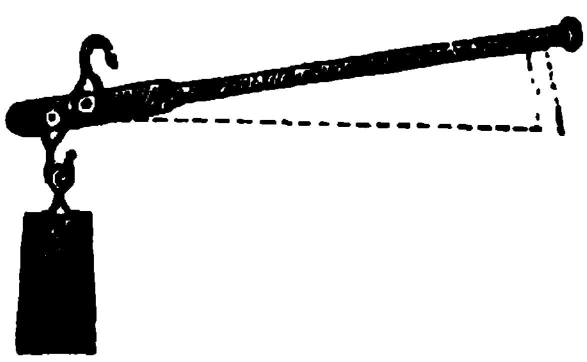
In a combination of levers, the power will be in equilibrio with the weight, when the former is to the latter, as the continued product of all the arms of the lever, on which the weights act, is to the continued product of all the arms on which the power acts.

144. Combinations of levers may, upon this principle, be used as weighing machines: for if the relation between the lengths of their several arms be known, the relation between a known weight, acting at one extremity of the first lever, to the unknown mass which is in equilibrio with it at the farthest point of the system, is also known; and the weight of the latter will be determinable. The most useful application of such a combination of levers is in the platform balance, of which the following is a description:



AB is a section of a platform of wood, of a rectangular shape, resting upon a frame represented at H and I. CP″, C′P″, are levers of the second class, having their fulcrums at C and C′. Of these there are four, diverging from the centre of the platform, in the direction of its semi-diagonals, to the four corners. At W and W, upon the two that are represented in the section, are pins, which, by a small motion, may be brought in contact with the platform, and thus may be made to raise it from the frame, and bear its weight with that of the articles to be weighed. The extremities of these four levers rest upon a bar at P″, which is supported at the point W″, by another lever of the second class DP′. The last is connected by a wire reaching from the extremity P′, of its longer arm, to the shorter arm of a lever of the first class; at the opposite end of which a scale, S, is suspended. A weight placed in the scale S, will raise the point P′, and with it the bar P″; and the rise of the latter will cause the weight of the platform, and the articles with which it is loaded, to press upon the pins W, W. Thus a small weight in the scale, S, will be in equilibrio with a large one on the platform AB, and their relation will be given by the formula (113). The lever, P′D, is generally placed in a position at right angles to that in which it is represented, projecting beyond one of the longer sides of the rectangle: and it is evident, from mere inspection, that the four levers, P″C, &c. act, so far as the conditions of equilibrium are concerned, as a single one. Four are used, in order to bear the platform at a sufficient number of points, and cause it to rise and fall parallel to itself.

145. A Steelyard is another modification of the first kind of lever, which is also used as a weighing machine. The lever in this case has, as represented in the figure on the next page, unequal arms. To the shorter of these, the substance to be weighed is attached, and



its weight is determined by means of a constant known weight, that is moved to different distances from the fulcrum, until it be in equilibrio with the substance to be weighed. If their distances from the fulcrum be equal, the two weights are equal; if the constant weight be twice as far from the fulcrum as the fixed point, whence the substance to be weighed is suspended, it will be obvious that the latter weighs twice as much as the former: and thus, as the distance of the constant weight varies in arithmetical progression, the unknown weight will vary as those distances. The longer arm of the steelyard is therefore cut into equal divisions, and the unknown weight is determined by the distance at which the constant moveable weight is from the fulcrum, at the time equilibrium takes place.

The constant weight is suspended from the lever, by means of a hook or ring that is cut beneath into a sharp edge, to enable it to adapt itself to the notches that form the divisions of the longer arm. Hence there is danger of these divisions being cut and widened, until the instrument ceases to give true indications. So, also, when the lever is inclined, the distance of the constant weight varies. In few cases, indeed, can the steelyard be depended upon for giving as true a measure of weight as the balance. If a scale be suspended by knife-edges, from the longer arm of an unequal lever, weights placed in it will have a value in determining the weight of a substance suspended from the shorter arm, as much greater than their true value, as the shorter arm is less than the longer. Upon this principle they may be graduated; and a weighing machine, thus constructed, will have an advantage over a balance when very great weights are to be determined; for the short arm will be less liable to break under their action than the arm of a balance, and the load upon the knife-edges will be much less.

146. Were there no resistance to the motion of the lever, it could be set in motion by the smallest addition either to the power or the weight; but in this case, as in all others, friction will interpose; and if the weight be set in motion by the power, the latter must first receive an addition to its intensity, when merely in equilibrio, which is equal to the moment of rotation of the friction; and in case it is desired that the weight shall set the power in motion, the former must first receive a similar addition, in order to be ready to cause motion by the smallest new accession of force.

In the lever, the general equation of equilibrium is

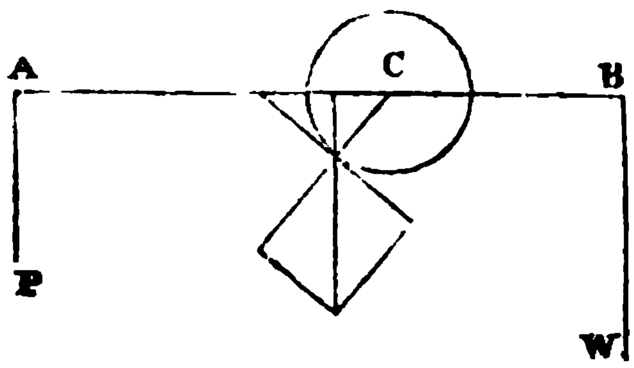
$$Pp = Ww;$$

if we let the resistance, whatever be its nature, $=R$, and let Rr be its moment of rotation, then in the case in which the system is ready to be set in motion,

$$Pp = Ww \mp Rr, \tag{114}$$

which will be a general condition in all machines whatsoever.

To apply this to the case of the lever:



Let AB be the lever acted upon by the parallel forces P and W, and turning upon the cylinder C, as an axle; call the arms p and w. Let the power, P, be on the point of setting the weight, W, in motion. The pressure on the axle is equal to the resultant of the two forces, P and W; resolve this resultant into two forces, one of which is a tangent to the axle, the other a normal. Let m be the angle the tangential force makes with the direction of the resultant, the two components are

$$(P+W) \cos. m,$$
$$(P+W) \sin. m;$$

that part of the friction which is directly opposed to the motion, will have the same ratio to the whole friction, f, as the latter of these components to the whole pressure, or will be

$$f(P+W) \sin m,$$

and
$$\sin. m = \frac{1}{\sqrt{(1+f^2)}};$$

whence the friction becomes

$$\frac{f(P+W)}{\sqrt{(1+f^2)}}; \tag{115}$$

and as r is its distance from the centre of motion, Rr in the general formula (114) becomes

$$Rr = \frac{fr(P+W)}{\sqrt{(1+f^2)}}; \tag{116}$$

and the condition of the state of the machine, in which motion is about to begin, is

$$Pp = Ww + \frac{fr(P+W)}{\sqrt{(1+f^2)}}. \tag{117}$$

The co-efficient, f, is in most cases, so small a fraction, that its square, f^2, may be neglected, in which case the formula becomes

$$Pp=Ww+fr(P+W). \qquad (118)$$

Of the Wheel and Axle.

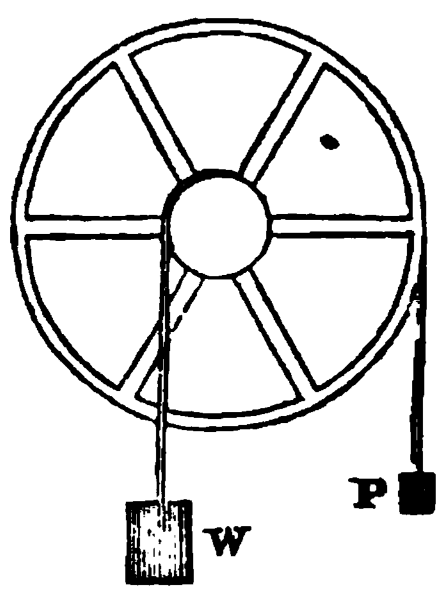
147. The Wheel and Axle, as its name imports, is a wheel firmly connected to an axle, and moving with it upon a common axis. The power is applied to the circumference of the wheel; the weight to the circumference of the axle. It may, therefore, be considered as a lever having its fulcrum in the axis, and the power, in a state of equilibrium, will be to the weight, as the radius of the axle to the radius of the wheel, or

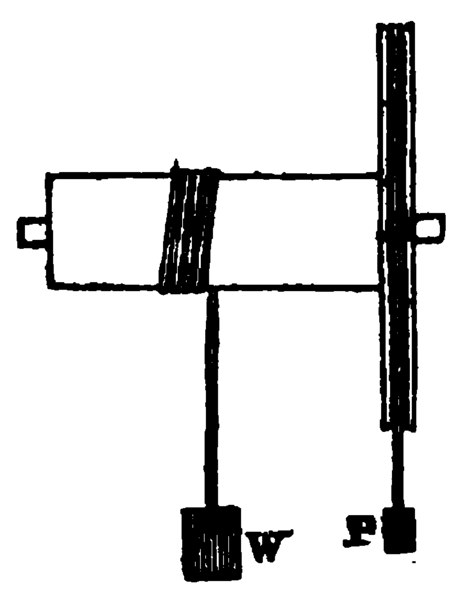
$$Pa=Wb. \qquad (119\,a)$$

But as the circumferences of wheels are proportioned to their radii, the latter part of the proportion may be changed; and when the wheel and axle is in equilibrio, the power is to the weight as the circumference of the axle to the circumference of the wheel, or

$$P.2\pi a=W.2\pi b. \qquad (119\,b)$$

The simplest form of the wheel and axle is represented beneath. In it the power is applied by an endless rope passing over





the circumference of the wheel, and the weight to a rope coiled or wound around the axle. Such is the form which is habitually used in our warehouses.

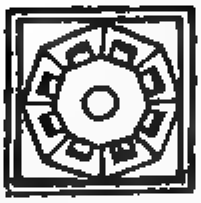
It is not necessary that the wheel should be continuous: a single spoke, or several, projecting from the axle, will be sufficient. The axis may be either horizontal or vertical; in the former case, an axle with bars or spokes, is called a Windlass; in the latter, a Capstan.

The windlass used in ships has a number of holes cut in the direction of its length, upon four different parts of its circumference; handspikes or bars are placed, when the engine is to be

used, in the row of holes that is uppermost, and men springing to these bars, act upon them partly by their muscular force, and partly by their weight. It is therefore an application of human force that requires great exertion, and produces corresponding effects for a short time.

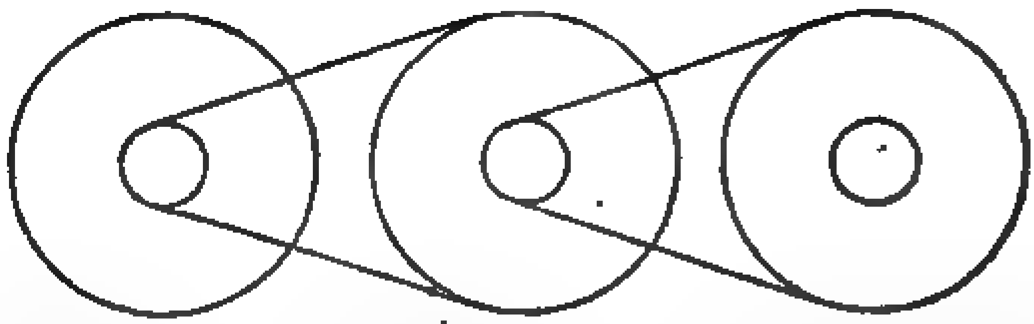
The windlass which is used by well-diggers, has a bar at each end; this bar is bent at right angles, in order to furnish a convenient handle to the persons that work it. A handle thus formed is called a *Winch,* and is of frequent application in many useful cases.

A capstan with a single bar, to which a horse is harnessed, is often used on the shores of our bays and rivers for the purpose of drawing up logs and balks. In a ship's capstan, a number of bars are inserted into the head, in the manner of the spokes of a



wheel. It is manœuvred by men placed between these bars and walking around. The exertion is therefore moderate, and can be long continued. In large ships, the capstan passes through the decks, and thus a gang of men may be applied to it on each deck. In all these cases the weight is applied to a rope wound around the axle.

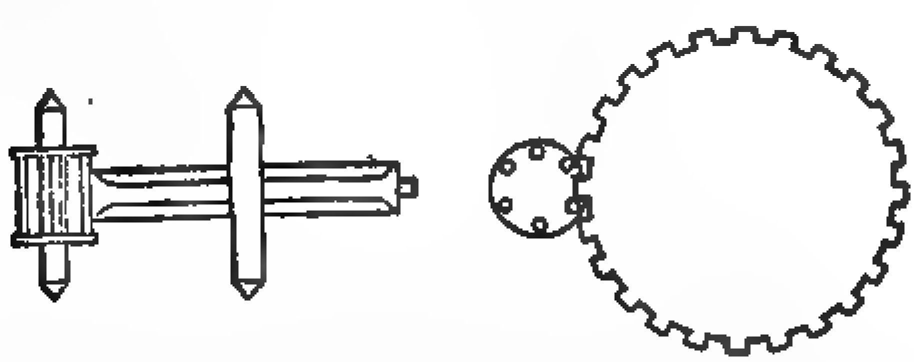
148. A series of wheels and axles may be combined together, as in the figure, by means of endless ropes or bands; one of these



is passed over the circumference of a wheel, and the circumference of the axle of the adjacent wheel; the number of bands in the system is always one less than the number of wheels. The action is identical with that of a system of levers. The power, therefore, will be in equilibrio with the weight, when the former is to the latter, as the continued product of the radii of all the axles is to the continued product of the radii of all the wheels.

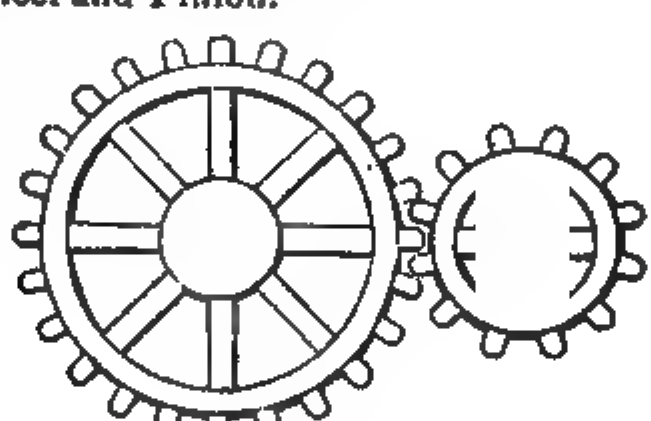
149. Wheels and axles may also be combined, by making them turn each other by the friction of their surfaces, and giving these such a form as to exert a direct pressure upon each other. For this purpose—

(1.) Projecting pieces or cogs, as in the figure, may be adapted



to the circumference of the wheel, and the axle of the next may be formed of two parallel circular plates, united by round staves, arranged in the circumference of a circle: such an arrangement is called a Cog-Wheel and Trundle: or,

(2.) The circumferences of both the wheels and axles may be cut into teeth, as in the figure beneath. Such a modification is called the Wheel and Pinion.



In conformity with the principles of a combination of levers, of which this is an obvious application, equilibrium will exist in a series of wheels and pinions, where the power is to the weight, as the continued product of the radii of all the pinions, to the continued product of the radii of all the wheels.

Were the radii of the pinions, and their number of teeth, exact aliquot parts of the radii of the wheels, and of their number of teeth, at each revolution of the wheel, the same tooth, upon its circumference, would fall between the same two teeth of the pinion. From this would arise an unequal wear. To prevent this, the number of teeth on the wheel ought to be such as is prime to the number of teeth upon the pinion. This is usually effected by adding one additional tooth to the wheels. A number of revolutions, then, equal to the number of teeth in the wheel, must take place before the same two teeth of the adjacent wheel and pinion can again come into contact. Such an additional tooth is called the Hunting Cog.

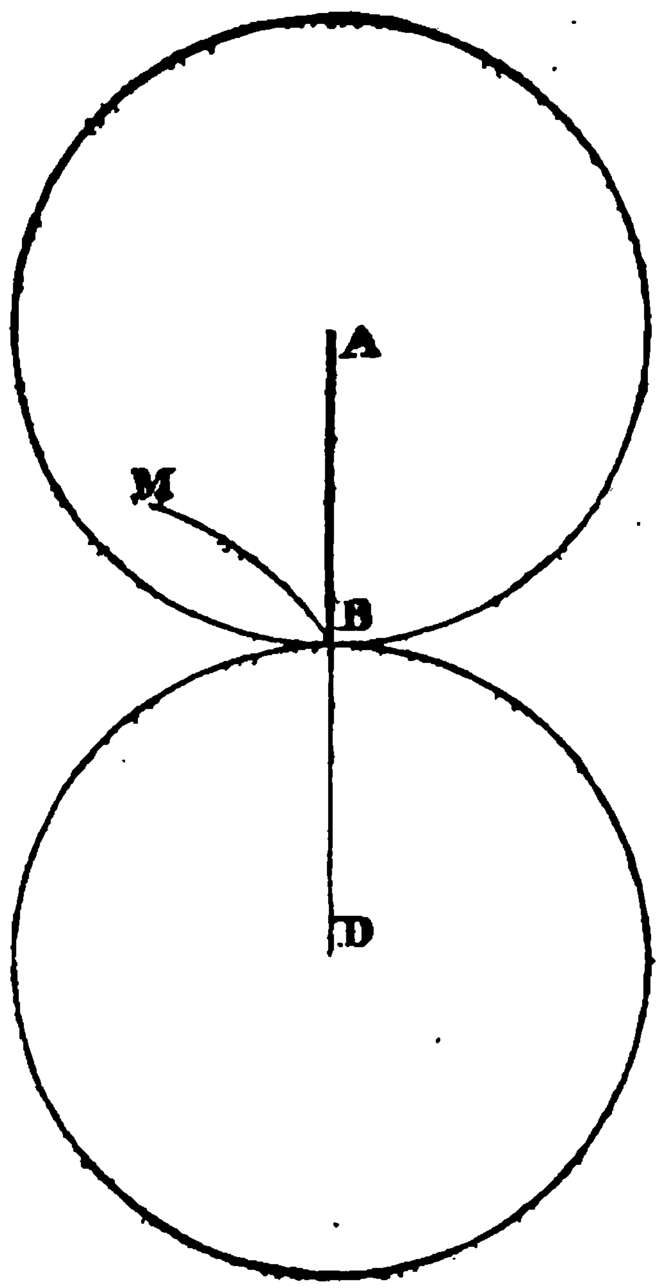
To express the conditions of equilibrium in terms of the number of teeth: The power must be to the weight as the continued product of the number of teeth on all the pinions is to the continued product of the teeth of all the wheels.

150. It is of great importance in systems of wheels and pinions, that the teeth should have a proper curvature, in order that the action of the power may be communicated to the weight as directly, and with as little friction as is possible. It would occupy too much space to enter fully into the detail of the construction of the teeth of wheels, but it is necessary that the general principles be explained.

We shall take the case of two wheels situated in the same plane.

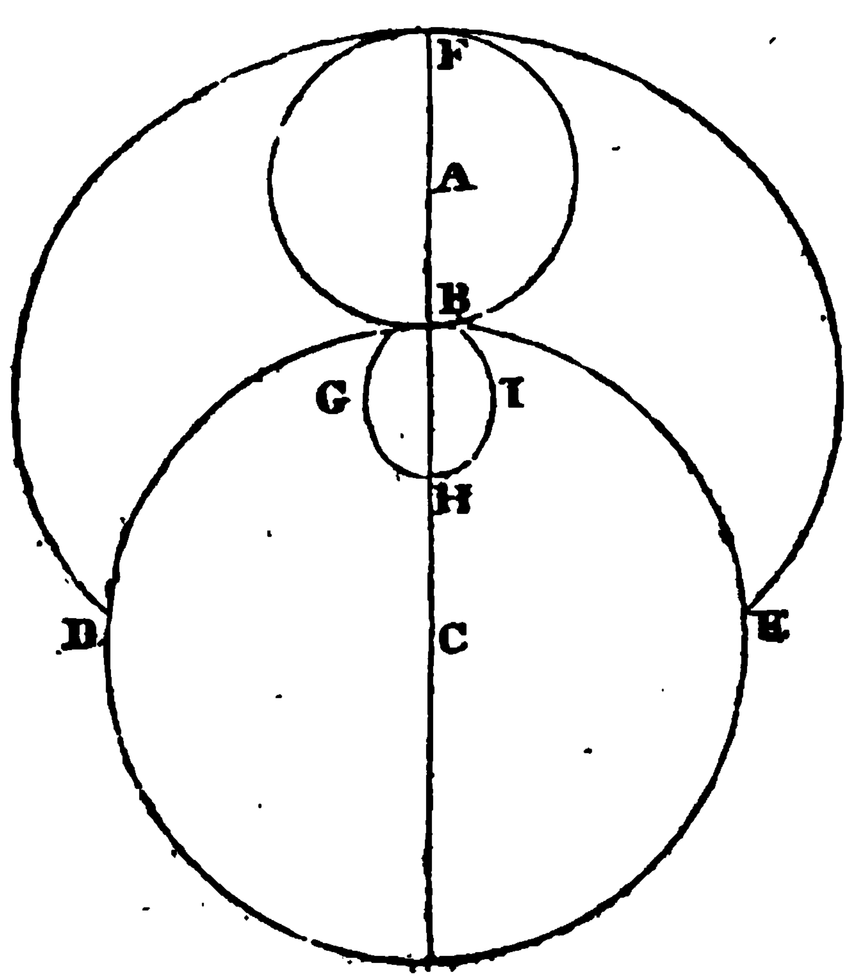
Let two circles touch each other, each being moveable around the centre. A constant force in the direction of the common tangent of the two circles, would make the circumference of each revolve with equal velocities. In order that one of these circles should transmit its motion to the other, simple friction might at first be sufficient, but this would speedily wear them away and thus disunite them. In practice, then, it is necessary, that for the two circles that touch each other, two others described around the same centres, and with diameters having the same ratio to each other, should be substituted, and that on their circumferences should be placed teeth. These teeth, in order to keep up an equable communication of motion, must satisfy this condition: that in the action of the teeth, which will be in the direction of the common normal of the surfaces in contact, the two primitive circles shall be moved as if they were propelled by a force in the direction of their common tangent.

Let there be two circles whose radii are AB, BD, that touch each other at the point B: let there be fixed upon one of the circles a tooth terminated by the curve BM, and on the other a radius AB, the tooth, in turning, will move the radius of the other circle; the condition to be fulfilled is that the curve BM shall, in all its positions, touch the radius, and the perpendicular to the radius BA, at the point of contact, shall always pass through the fixed point B.



The curve which will satisfy the conditions, is an epicycloid, formed upon each of the circles by a point in the other, supposing the latter to move upon the circumference of the former, in the same manner that a circle moves upon a line when the common cycloid is generated.

Thus in the figure, if the circle, whose radius is AB, be sup-



posed to move upon the circumference of the circle DBF, the point F will describe an epicycloid DFE, and some portion of this curve will be the proper figure to give to the projection of the teeth.

The spaces between the teeth are formed by supposing the radius of either of the circles to be produced; and that the point on its extremity, shall, while the epicycloid is described, describe a curve such as GHI; this will give a form proper for the interval of the teeth. The two curves cannot be made to unite except at an angle; hence it is necessary to join them by a straight line, which is a common tangent.

If the teeth of one of the wheels have their forms determined, the teeth of the other will no longer have the same form, but must be modified so as to adapt themselves to the teeth of the first, in conformity with the condition we have stated.

Thus in the case of the wheel and trundle, the staves of the latter may be considered as teeth, whose sections are circular. In this case, the cavities between the teeth will be portions of a circle whose radius is the same as that of the staves of the trundle; the projections will be curves parallel to the epicycloid described upon the circle whose radius is BD, by the circle whose radius is AB.

However regular the curves may be, and however completely they may satisfy the prescribed condition, there will be an inequality in their pressure on each other. This inequality may be lessened by increasing the number, and lessening the size of the teeth. In cases where intensity of force is gained at the expense of velocity, this inequality becomes less and less perceptible at each addition of a wheel and pinion to the system; but in the case where velocity is gained, the apparent inequality is multiplied in exact proportion to the increase of velocity.

In the former case, the successive variations will be repeated several times within the time in which two teeth are in contact; in the latter, they will be found to affect several teeth, each of which will be unequally impelled.

151. The circumferences of the wheel and pinion that are in contact will move, as has been seen, with equal velocities; their moments of rotation are therefore proportioned to their respective diameters. Hence, when wheels drive pinions, the intensity of the force is diminished, and when pinions drive wheels, increased. In the former case velocity is gained, in the latter it is lost.

152. When a motion is to be changed so that its direction shall lie in a different plane from that in which the forces had before acted, the wheel and pinion offers various modes of effecting the change. Thus: the teeth may be cut upon the surface of a hollow cylinder, and stand at right angles to the plane in which the

wheel revolves; the axis of the pinion must then be perpendicular to the axis of the wheel, and the motion will be taken off at right angles; such a wheel is called a Contrate Wheel. The same may be effected by forming the teeth upon the surfaces of two conic frusta, whose axes meet, and make with each other an angle, which is the supplement of the required change of motion. The figure of the teeth, in this case, is derived from a curve called the spheric epicycloid. Such wheels are called Bevelled, or Mitre geering, according as the angle their planes make with each other, is right or oblique.

Let VAB, and VAD be sections of right cones; wheels and pinions constructed upon their frusta, will have the same me-

Fig. 1.

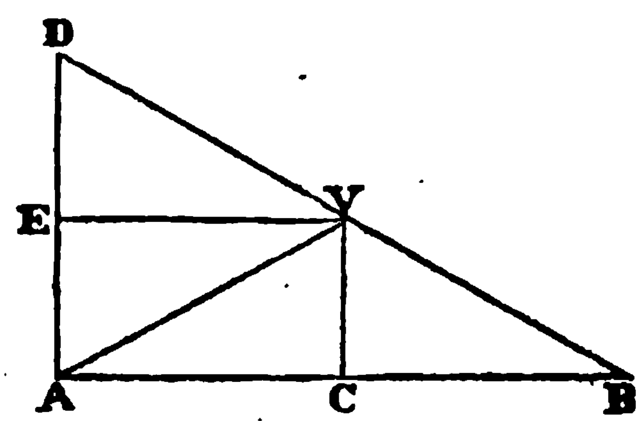
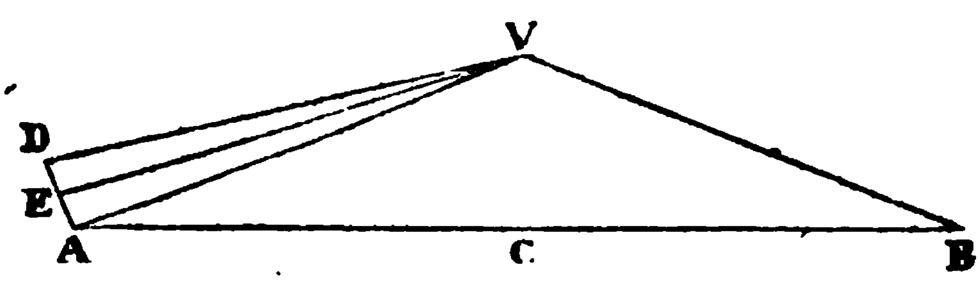
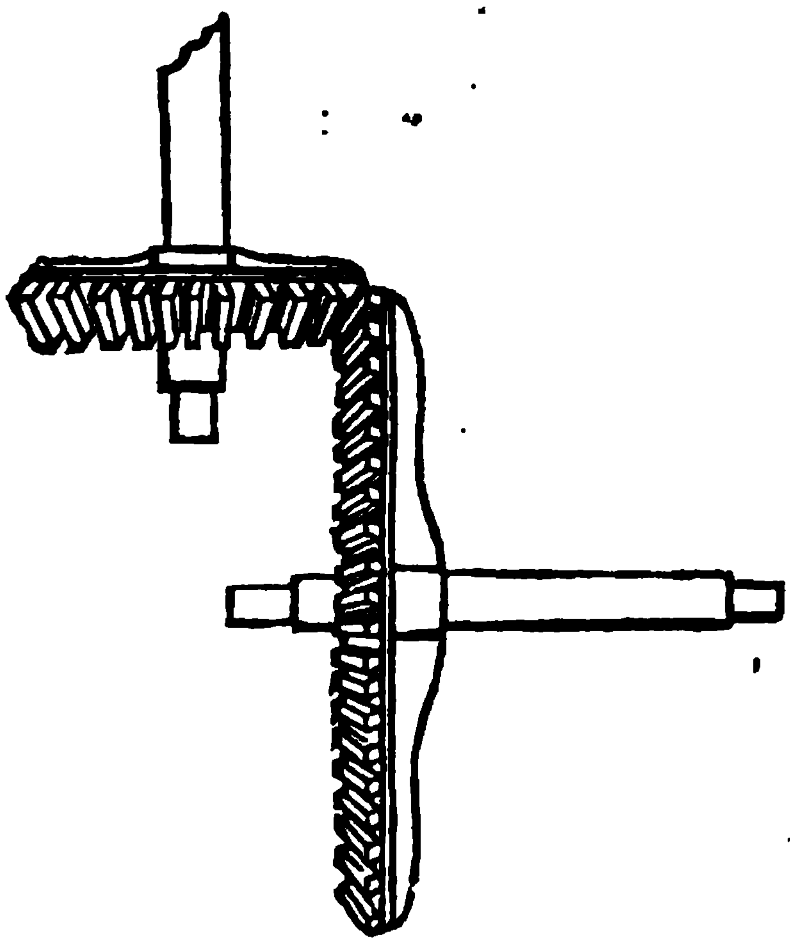


Fig. 2.



chanical properties as if they were in one plane. They will besides change the direction of motion to a plane, making an angle with that in which the original motion is performed, equal to the complement of the angle made by their respective axes, VE, and VC. In Fig. 1st, the motion is taken off at a right, in Fig. 2d, at an obtuse angle.

In the figure beneath, a pair of mitre wheels, with their teeth entering into each other, is represented.



Such wheels are sometimes equal in size, and answer no other purpose than to change the direction of the motion.

153. In the original and simpler forms of the wheel and axle, the friction of the pivots is estimated exactly as we have done it in the case of the lever: but before the machine can be ready to be set in motion, an additional force must be applied to overcome the resistance of the rope or ropes. In the capstan and windlass there is but one rope, and the equation of final equilibrium, when the smallest force added to the power, will cause motion to begin, is

$$Pa = Wb + (P+W)\,fr + b\,(m + p\,W)\,; \qquad (119c)$$

in which expression, a is the radius of the wheel, b that of the axle, and r of the gudgeon on which the motions are performed. Coulomb, in order to give an instance of the application of his theory, and the results of his experiments to practice, calculates the joint resistances of the friction, and the rigidity of ropes, in the case of raising 8000 lbs. by means of a capstan, and finds that one tenth part of the moving power must be expended upon these resistances.

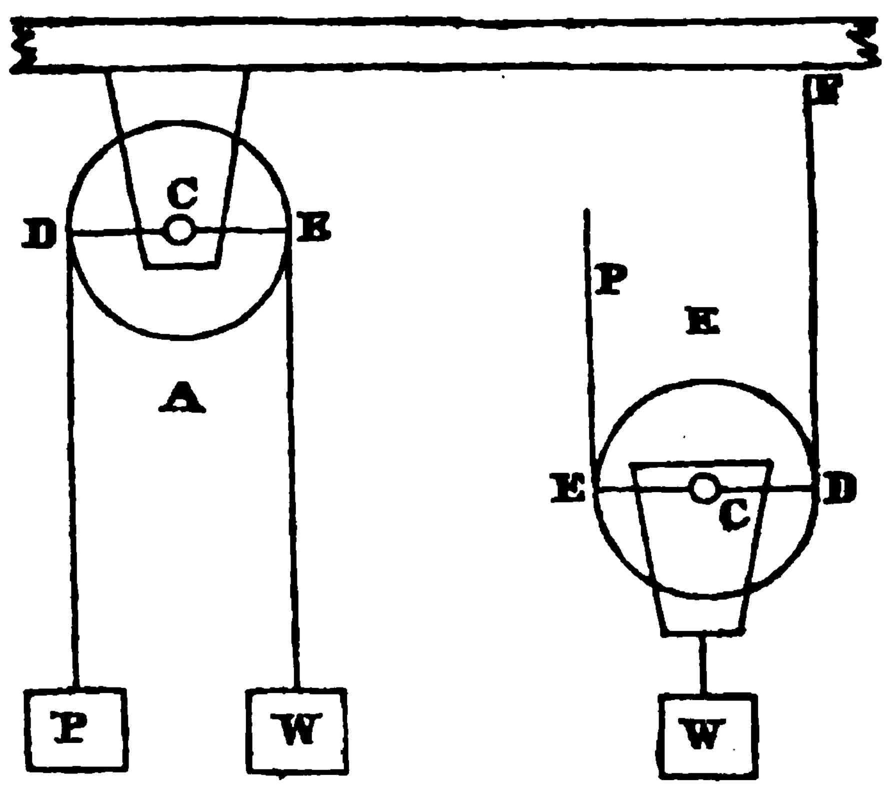
154. In the case of a wheel acting upon a pinion, or upon a trundle, we shall not enter into a full investigation of the friction: the old practical rule of allowing one eighteenth part of the power, applied to its own arm of the lever into which the wheel may be resolved, having been found to correspond with the results of theory. A similar allowance must be made for every increase in the

number of wheels and pinions in the system; the additional friction being, in each, one eighteenth part of the force the wheel exerts upon the pinion it drives.

Of the Pulley.

155. A Pulley is a wheel, moveable upon an axis, and having a groove cut upon its circumference, over which a cord passes. It is enclosed in a box or case that supports the axle, which is called its Block. The block may be either fixed to a firm support, or moveable. In both cases, the power is applied to one end of the rope; in the case of the fixed pulley No. 1, the weight

No. 1. No. 2.



is applied to the other extremity of the rope, in the moveable pulley No. 2, the weight is suspended from the box or block.

In the fixed pulley, No. 1, the direction of the motion is alone changed, for the power and weight have equal moments of rotation, and hence may be considered as acting upon the equal arms DC, CE, of a lever of the first kind, hence

$$P=W. \qquad (120)$$

In the moveable pulley No. 2, the rope is fastened at one end to the fixed support, F; this may also be considered as a lever, but the fulcrum is not at the centre, but at the point D, hence

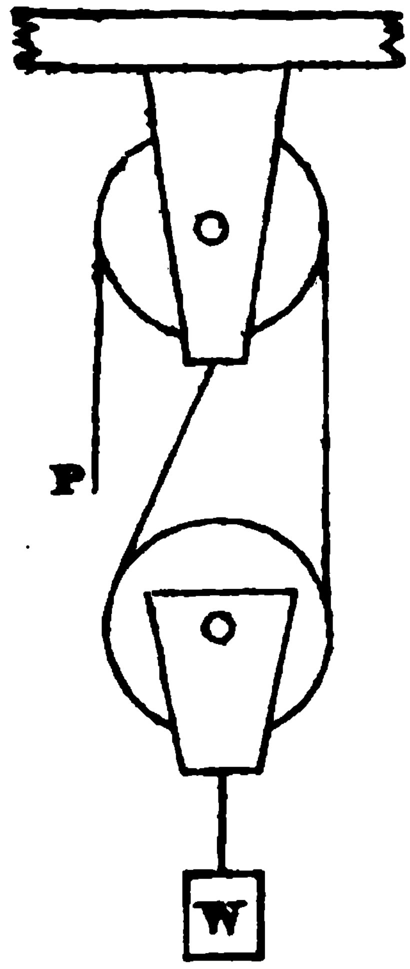
$$P : W :: DC : DE;$$

but one of these lines being the radius, the other the diameter of the same circle,

$$P=\tfrac{1}{2} W. \qquad (121)$$

In the moveable pulley, the direction of the force P is not changed. But it is frequently desirable that the intensity of the power shall not only be doubled, but that its direction shall

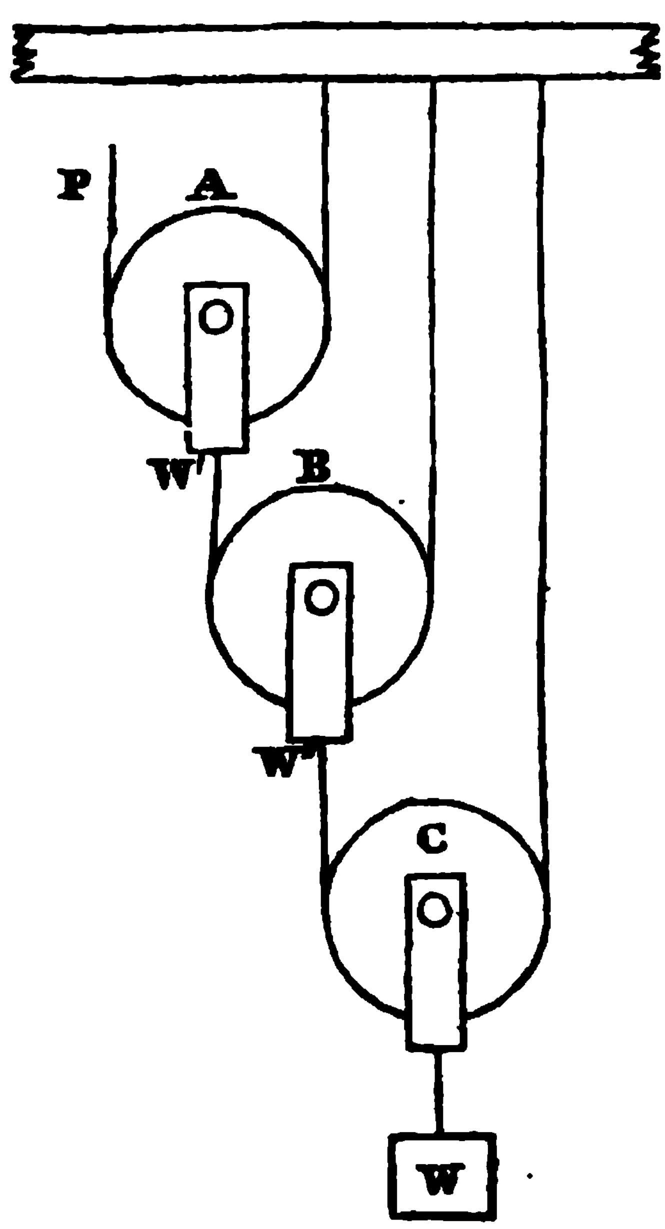
be changed. In order to effect this change, a fixed pulley may be combined with a moveable pulley, as in the figure.



Here the condition of equilibrium is still the same, or

$$P=\tfrac{1}{2}W.$$

156. Pullies, whether fixed or moveable, may be combined together in various manners; thus, as in the system beneath; for the weight that acts upon the moveable pulley A, may be substituted a rope that is wound around a second pulley, B; to this, in like manner, a rope passing over a third pulley, C, may be applied, and the weight may be attached to the box of C; the pulley, A. doubles the intensity of the power, or



$$P=\tfrac{1}{2}W';$$

the pulley B does the same to the force which acts upon it, which is W'; and hence

$$W'=\tfrac{1}{2}W''.$$

The pulley, C produces the same change in the force W'', therefore,

$$W''=\tfrac{1}{2}W;$$

whence we obtain

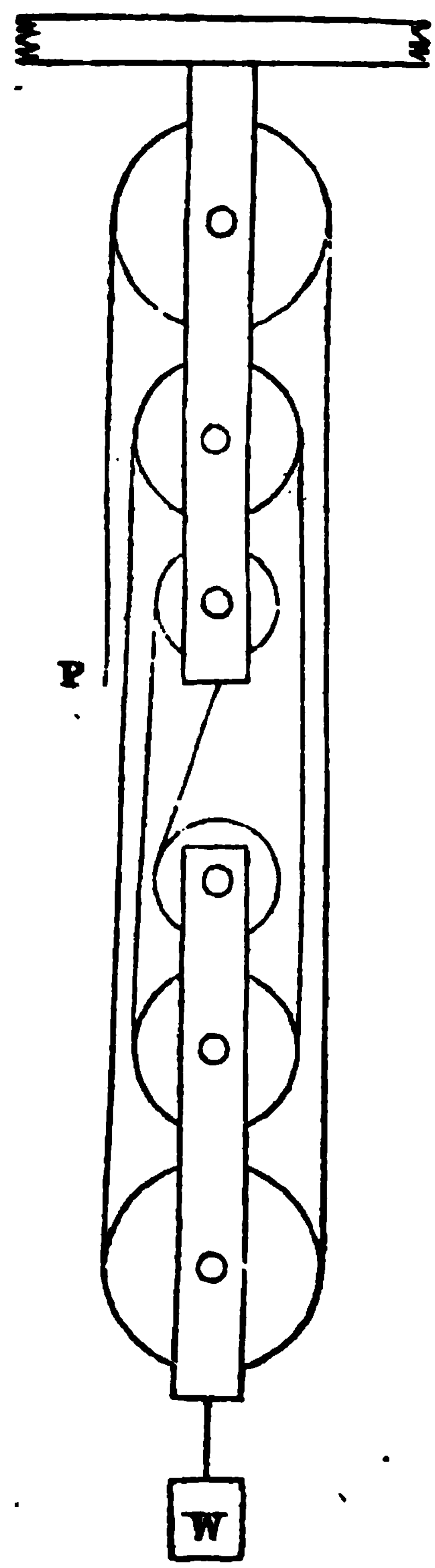
$$P=\tfrac{1}{8}W.$$

It will be obvious, that in such a system, the intensity of the power is increased in a geometric progression, whose common ratio is

2, and whose number of terms is the number of moveable pullies. Hence, in a system whose number of moveable pullies is n,

$$2^n (P) = W. \qquad (122)$$

This mode of combining pullies, is not convenient in practice, and hence, in spite of its great power, it is not often used. A system in which the number of fixed and moveable pullies is equal, and all the moveable and all the fixed pullies are combined, each kind in a single block, although it causes a less increase in the intensity of the power, is, on account of its great convenience, in much more extensive use. Such a system, composed of three fixed and three moveable pullies, is represented below. It will



be easily seen that as the rope is now continuous, the intensity of the force cannot go on increasing in a geometric ratio. To determine the condition of equilibrium: the weight is supported, in this case, by six ropes, or rather six separate parts of the single rope; each of them undergoes a tension due to the force P; they, by their united effort, support the weight, which is, therefore, the resultant of these tensions, or of six equal and parallel forces; hence

$$P=\tfrac{1}{6}W,$$

and for any number (n) of moveable pullies,

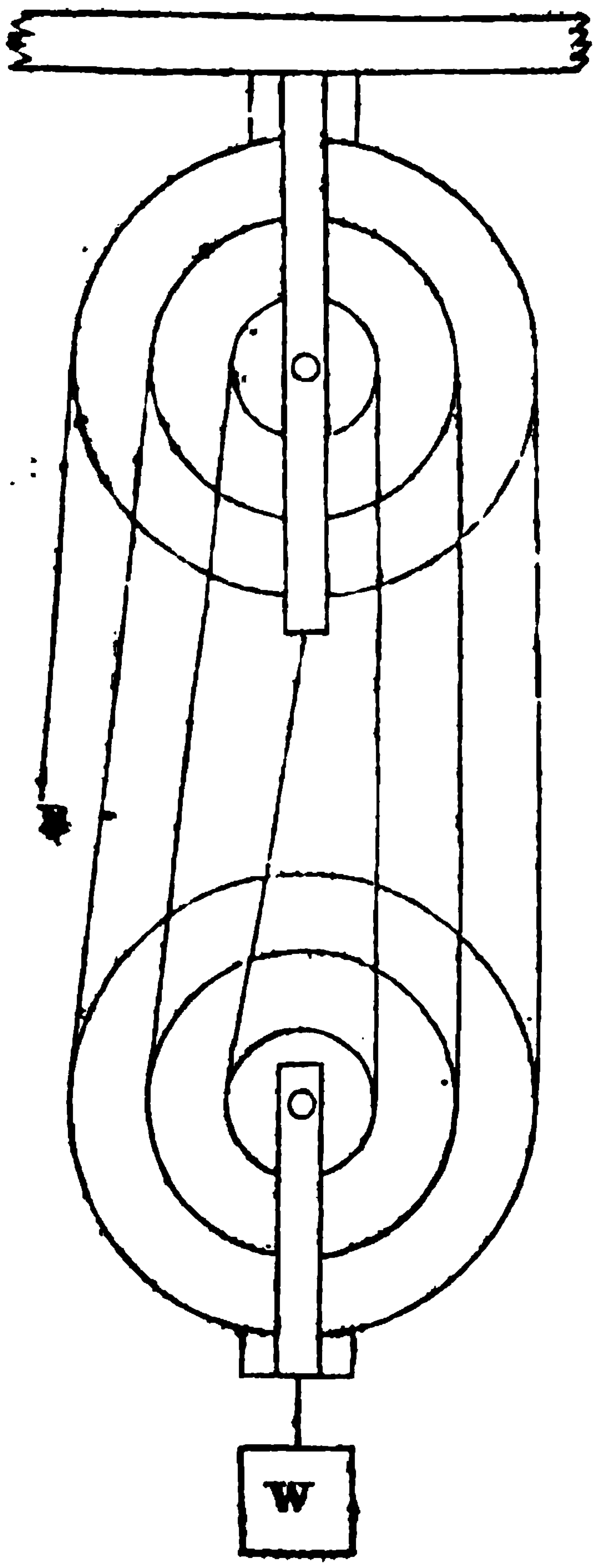
$$2n\,P=W,$$

$$\text{and } P=\frac{W}{2n}. \qquad (123)$$

A similar system may be formed by placing all the pullies of each kind, in a box in such a manner, that their axis may be the same. Their axles, however, must be separate; for they will not all revolve in equal times. The pair, composed of one fixed and one moveable pulley, nearest to the power, having, at its circumference, the same virtual velocity with the power; while the pair nearest to the weight has a virtual velocity no more than twice that of the weight. The velocities of the several pairs of pullies will, therefore, be as the series of natural numbers, the last term of which is the number of moveable pullies.

Out of this varying velocity grows an unequal wear upon the different axles, and the evil would not be diminished by making the pullies revolve at the same rate; for this would be in fact impracticable, without a vast increase of friction growing out of the dragging of the rope over circumferences having naturally different velocities, but which would be thus constrained to move at the same rate by their connexion.

These defects are obviated in the blocks of White. In this, each set of pullies is turned out of a single piece: the concentric circles in the figure on the following page, are the projections of the pullies, and are in an order of size corresponding to the series of natural numbers. Hence a common rotation upon the same fixed axis may be given to each set of pullies, which will have the same velocity with the ropes that pass over them, and a single axle to each block will be sufficient.



The modes in which pullies may be combined, may be varied almost infinitely: in them all, however, the same principles are applicable. It is not necessary then that we should pursue their modifications to any greater extent.

157. When the ropes are not parallel to each other, the direction of the forces becomes oblique, and thus the effect of a pulley or system of pullies, will be changed. The action of the power and weight may, however, be determined in all cases where the angles the directions of the ropes make with each other is known, by means of the theorems of the Composition and Resolution of Forces § 12.

158. The effect of friction upon pullies, and of the resistance of ropes, may be calculated upon the principles that have already been laid down, in the case of the wheel and axle. In a single pulley, the equation of the state in which the smallest additional force will cause motion, is the same as in the wheel and axle, except that the radius of the gudgeon r in (119c) becomes the same as b, and the formula is

$$Pa = Wb + (P+W)fb + b(m+pW). \qquad (124)$$

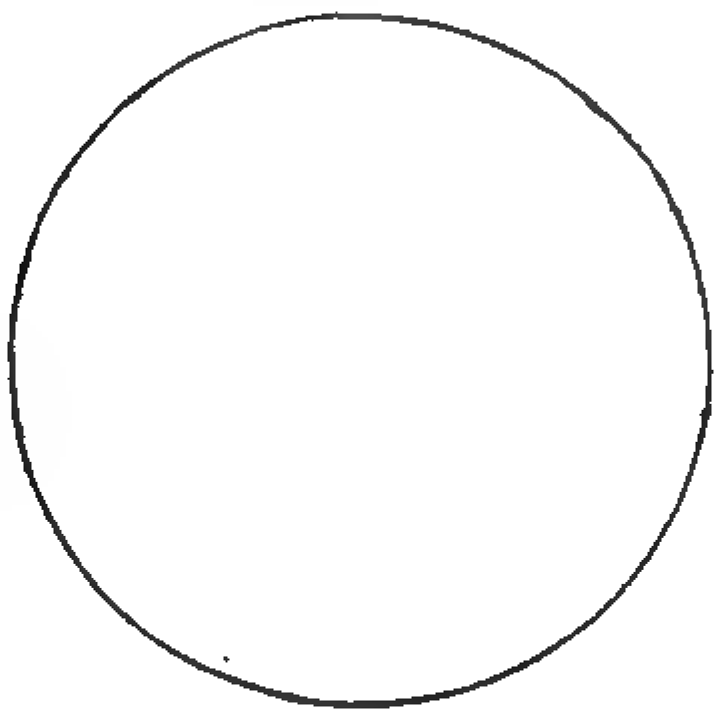
In a complex system of pullies, this becomes difficult of application. If, however, we assume, 1. That all the ropes are parallel; 2. That all the wheels and all their axles are equal in diameter; 3. That the function f, which represents the proportion of the friction to the pressure, is very small; 4. That the resistance of the rope is proportioned to its tension, and that hence, in formula (124) $m=0$. Upon these assumptions, making n=the number of moveable pullies. and

$$\frac{a\,(p+c)+fb}{a-fb} = A,$$

the expression for the state in which the smallest addition to the power will cause motion, may be found to be

$$P=W.\left(\frac{A^n\,(A-1)}{A^n-1.}\right) \qquad (128$$

The friction of pullies has been considerably lessened by an application of friction wheels, (see § 125,) made by the late Mr. Garnett, of Brunswick, New-Jersey. Their plan is represented beneath.



The axle of the pulley, it will be seen, rests upon six wheels, enclosed in a box, and the friction is diminished in the ratio of the diameters of these wheels to the diameters of their axles.

Of the Wedge.

159. The Wedge is a triangular prism, of some hard material, whose section is, generally speaking, isosceles. The power is applied perpendicularly to the surface on which it acts; the weight is a resistance which is resolved into two parts, each of which acts perpendicularly upon the other two faces of the wedge. In order that equilibrium shall exist, these forces must, § 14, converge to a point within the wedge, and must be to each other in the ratio of the sides of the triangular section on which they act.

It will be at once seen from § 14, that three oblique forces can only be in equilibrio when they converge to a point, and three such forces are proportioned in magnitude to the three sides of a triangle formed by lines drawn perpendicular to the direction of the forces.

160. In an isosceles wedge, the weight is applied to the two equal sides, the third side may be called the head of the wedge; and equilibrium will exist when the power is to the weight as the thickness of the head of the wedge is to twice the length of either of its sides. This is an obvious inference from the general proposition, and has the form of the following equation, a being the thickness of the wedge, and b the length of either of its sides.

$$P=\frac{Wa}{2b}. \qquad (126)$$

161. The theory of the equilibrium of the wedge is of little use in practical mechanics, for it is generally used in splitting or cleaving, and is rarely, or never, acted upon by the constant application of a force, but is impelled by a moving body striking against it at intervals, or as it is styled, by percussion. The effects, in this case, are proportioned to the weight of the moving body multiplied by the square of its velocity. For the aggregation of the particles of the body into which it is thus driven, may be considered as a constant retarding force, and from (61b)

$$s=\frac{a^2}{2g}:$$

it therefore appears, that the distance to which a body, retarded by a constant force, will go, before it loses its whole velocity, is proportioned to the square of its velocity; hence the striking body will continue to impel the wedge, until the latter has produced an action proportioned to the weight of the former, multiplied by the square of its velocity.

The effect of the wedge is still farther increased by the agitation produced by collision, among the particles of the body into which it is driven; they are by this action more easily penetrated, as is obvious from the fact that repeated blows will destroy the aggregation of the strongest substances. This total destruction of aggregation is, however, only finally effected by numerous shocks, unless they be of great intensity; and, generally speaking, the force of aggregation becomes, so soon as the earlier blows cease to be felt, as strong, to all appearance, as it was at first. Hence, in the act of splitting or cleaving bodies by a wedge, the body closes forcibly upon the wedge at the instant the striking body ceases to act, and by its pressure produces a friction, sufficiently great to retain the wedge in the position to which it has been driven. This great friction adds another most important property to the wedge, namely: that, although impelled by an intermitting force, it does not return back to its original position, when that force ceases to act, but retains all

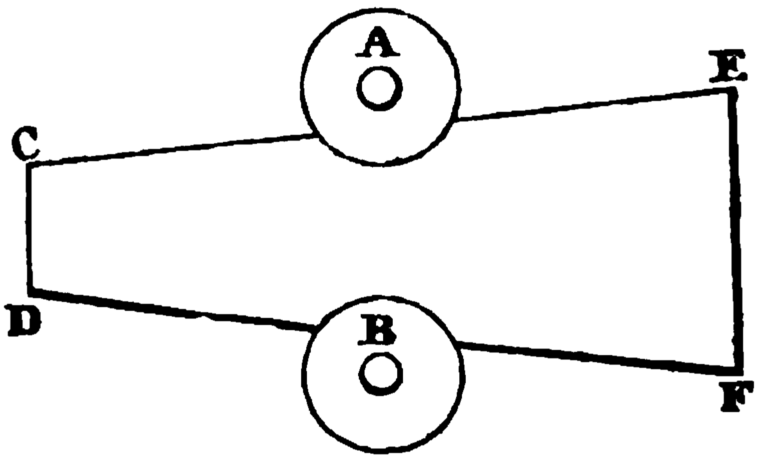
the advantage derived from previous impulses, to the whole of which any new impulse is superadded.

162. The valuable applications of the wedge are:

(1.) To all splitting and cleaving instruments, and to every variety of edge tools.

(2.) To obtain great pressures by means of small forces, wedges being driven between a firm obstacle and the body to be compressed.

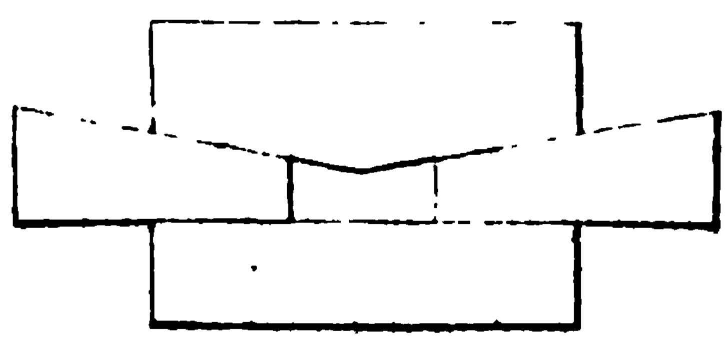
A printing press by Rust, of New-York, is also an application of the wedge. The two rollers, A and B, fall into cavities



of the wedge CDEF, and are fixed, one to the upper part of the frame, the other to the platten. When the press is to be set in action, the small end, C.D, of the wedge is drawn out by a combination of levers, the rollers then slide along the plane faces of the wedge, and a great pressure is produced.

(3.) To raise great weights to a small height: each successive impulse applied to the wedge raises the weight a small distance, whence it does not again fall; for the friction retains the wedge in its place, until a new blow be struck.

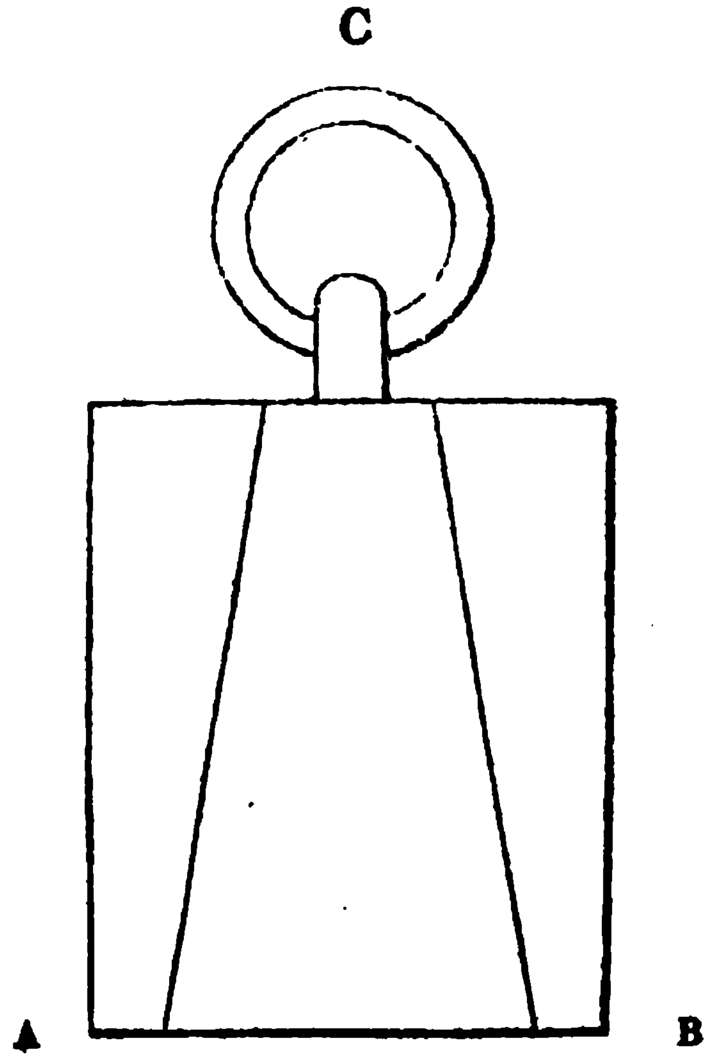
One of the most valuable and beautiful applications of the wedge, is to the lifting blocks of Seppings, by means of which the vast weights of the largest ships can be raised and supported, when placed in dock for repair. A section of these blocks is represented beneath.



A modification of these blocks, which is said to be even more convenient in use, has been invented by Thomas, an engineer in the employ of the American Navy Department.

(4.) There is an application of the wedge that is called the Lewis, which is employed in speedily attaching great weights to

the engines that are employed to move them. In a large stone, for instance, a hole of the figure of a truncated cone, that does not differ much from a cylinder, is cut, the larger base forming its bottom; into this is dropped an instrument of the following form. A cylinder is cut by two planes equally inclined to a plane passing through its axis, and when the three pieces are laid together, the whole has the exact cylindrical form. A section of this instrument is represented beneath; the side AB is placed lowest, and a force applied to the ring C; the effect of this upon the



wedge-formed piece, will be, to force the outer pieces against the sides of the cavity, and the pressure thus produced, will cause so much friction as to prevent the apparatus from being withdrawn by any force not sufficient to break the substance in which the hole is cut. By this simple and ingenious instrument, stones of several tons in weight may be firmly and suddenly attached to engines employed to raise them, and as instantly detached, when their weight is supported, and no longer acts upon the ring.

163. The principle of the wedge may be applied to the case of pressures upon its triangular bases as well as upon its inclined faces, and to bodies of pyramidal and conical figures: in all of these, the same effects are produced by a succession of blows, as in the simple triangular prism; and friction acts in the same manner to prevent their being withdrawn. Of this form, we find applications in all piercing tools, and in nails, spikes, treenails, and other similar instruments; they are employed for uniting the parts of such bodies as permit them to penetrate without any

great difficulty, when driven by successive blows, yet which close and retain them in their place, by means of great friction.

164. It will be obvious from what has been stated in speaking of the applications of this mechanic power, that friction, although it always resists the action of any power whatever, upon machines formed of such materials as nature furnishes, is not, on that account, an absolute loss. Indeed, few of the mechanic powers could act, were there no friction; the friction of the ropes, in the wheel and axle, and in the pulley, causes the cylinders on which they rest, to turn; and all the most valuable applications of the wedge, derive the principal part of their usefulness from the action of friction. So also it will be seen, that one of the two remaining powers would be of little use, were it not for its friction.

The parts of machines, of buildings, and other mechanical structures, could not be held together were it not for friction; and we shall find instances in which systems, that would otherwise be in a state of tottering equilibrium, are rendered stable by friction.

165. The friction which attends the use of the most valuable applications of the wedge, is not such as can be reduced to calculation, nor indeed is it important that it should.

Of the Inclined Plane.

166. The Inclined Plane is an instrument formed of a plane surface, in any position whatsoever, except parallel or perpendicular to the horizon. A body, placed upon such a surface, is actuated by its own weight, exerted in a direction perpendicular to the horizon, and which, being a constant force, would cause it to descend, as shown in § 47, with uniformly accelerated velocity; its descent is retarded by a constant force consisting in the resistance of the plane. These two forces, as may be deduced from § 56, have the ratio of the length of the plane to the length of its base, or of the cosine of the plane's inclination to the horizon, to unity, or

$$\frac{W}{R} = \cos.\ i.$$

The force with which the body tends to descend the plane, may be represented by $W \sin.\ i$; and as the power, if applied in a direction parallel to the line in the surface of the plane, in which the body would tend to descend, must, in order to cause equilibrium, be equal to the force with which the body tends to descend

$$P = W \sin.\ i. \qquad (127)$$

The sine of the angle of inclination is the ratio between the length of the plane and its height, hence:

167. In an inclined plane, the power, when in equilibrio with the weight, must have to it the ratio of the height of the plane to its length. The inclined plane, in this case, may be considered as a wedge in equilibrio, under the action of three forces, the relation between two of which, will determine the conditions of equilibrium.

But a weight may be supported upon an inclined plane, when the direction of the power is not parallel to the plane. In this case, the power must be resolved into two forces, one parallel to the surface of the plane, the other perpendicular to it, the latter has no effect, and the support will be wholly due to the former of the two components. By § 13, the value of this component will be, calling the angle the direction the power makes with the surface of the plane, a,

$$P \cos. a;$$

and the condition of equilibrium will be represented by the analogy

$$P \cos. a : W :: h : l, \tag{128}$$

in which h is the height, and l the length of the plane; and therefore,

$$P \cos. a = W \sin. i.$$

If the former act parallel to the base

$$a = i,$$

and

$$P = W \tan. i; \tag{129}$$

if we call the length of the base b, we have

$$\frac{h}{b} = \tan. i;$$

therefore,

$$P : W :: h : b; \text{ or} \tag{130}$$

when the power acts parallel to the horizontal base of the plane, equilibrium will exist when the power is to the weight as the height of the plane to the length of the base.

168. The inclined plane is a mechanic power of a frequent, nay, of almost constant application, in cases almost too numerous to be recited. Thus: we take advantage of natural slopes to raise bodies upon them, with powers of less intensity than they would require if raised vertically. We lift weights into wheel carriages by temporary inclined planes, formed of parts, either fixed or moveable, with which they are furnished. In raising

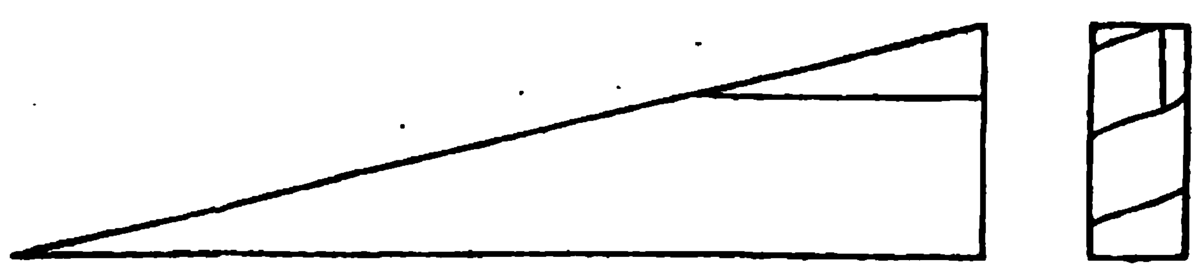
weights from cellars, or from one story of a building to another, we convert the stairs into inclined planes, by laying skids upon them. In all these cases, upon the principles in § 125, we save friction by making the body roll instead of sliding. We may also combine this mechanic power, in the case of rolling bodies, with a modification of the moveable pulley: thus, when a barrel is to be raised from a cellar, after laying skids upon the steps, and thus converting them into an inclined plane, we take a couple of ropes, and making them fast at the top of the plane, pass them around the barrel, and bring them back again to the top of the plane: a force applied to these ropes causes the barrel to revolve like a roller, and the intensity of the power, in addition to the gain it acquires from the plane, is doubled by the action of this temporary pulley. Inclined planes are also frequently constructed expressly for the purpose of enabling a power to counterbalance a weight of greater intensity: thus in saw-mills, the logs are drawn to the place where they are to be sawn, upon a fixed inclined plane. This principle has also been applied in the cases of roads, railways, and canals. In Morton's Marine Railway, vessels of great size are drawn from the water upon an inclined plane, by a power exerted through the intervention of modifications and combinations of the wheel and axle.

169. The friction of the inclined plane is easily determined, for it is always a function of the pressure; and the state in which the smallest addition to the power will cause motion, may be represented by the formula,

$$P=W \sin. i+f \cos. i. \tag{131}$$

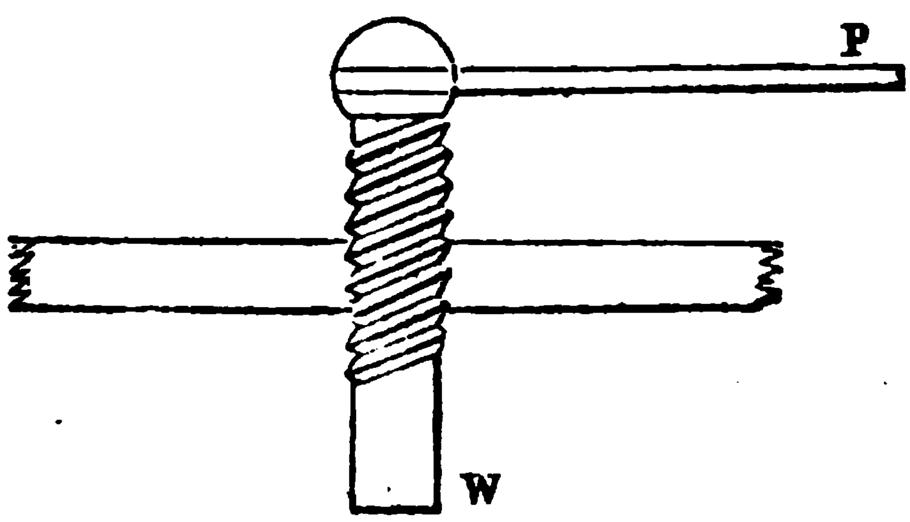
Of the Screw.

170. The Screw is a mechanic power that may be considered to be formed, as in the figure, by wrapping an inclined plane



around a cylinder: it is composed of a spiral ridge or thread upon the surface of a cylinder, which cuts every line that can be drawn upon its surface, and parallel to its axis, at an equal angle. The weight is applied to the extremity of the cylinder, and the power acts to turn the screw around; thus tending to propel the weight in the direction of the axis of the screw; it therefore becomes necessary that the screw shall move in a cavity to which

its thread adapts itself, or in a screw formed upon the surface of a hollow cylinder, which the solid screw exactly fills. The weight may be applied to the circumference of the cylinder, in the direction of a tangent to this circle. The screw, in this case, is an inclined plane, whose height is the distance between the convolutions of the thread, or, as we usually style it, the distance between the thread; for although there be but a single thread wound around the cylinder, yet, as when we view it, we see numerous convolutions, we call each of them a thread, as if they were actually separate. The base of this inclined plane is the circumference of the screw, hence from § 167, the power is to the weight, when in equilibrio, as the distance between the threads is to the whole circumference of the screw. It is far more usual to apply



the power, as in the figure, to a lever at right angles to the axis of the screw, or to a circular head, to whose plane the axis of the screw is a normal. In this case the condition of equilibrium is, that the power be to the weight, as the distance between the threads of the screw is to the whole circumference described by the point to which the power is applied.

Call the distance between the threads d, the circumference of the screw c; call the intensity of the force exerted by the power at the circumference of the screw, W′.

By the principle of the wheel and axle,

$$P : W' :: c : C;$$

whence

$$W' = \frac{PC}{c}.$$

By the principle just laid down for the screw,

$$W' : W :: d : c;$$

and

$$W' = \frac{Wd}{c};$$

whence

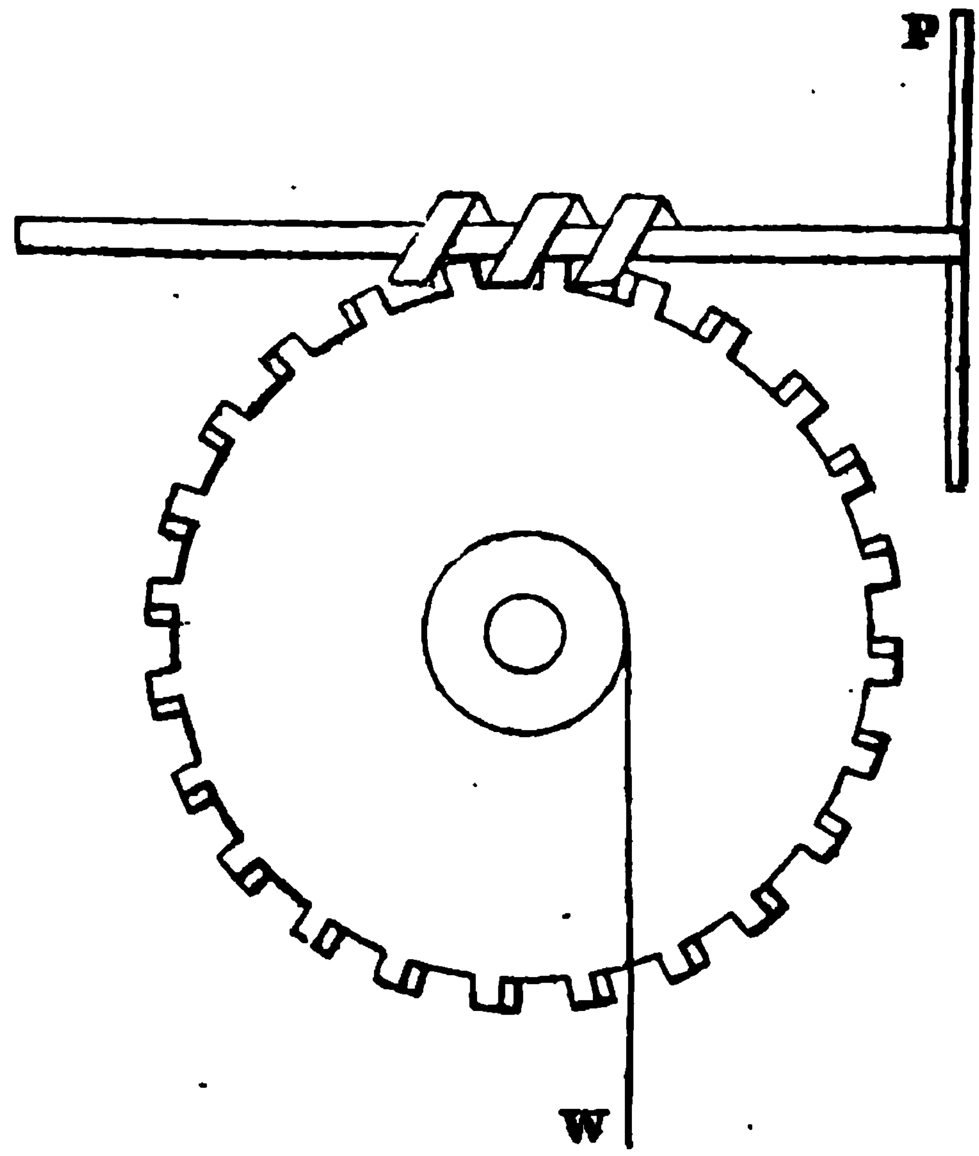
and

$$\left.\begin{array}{l} Wd = PC, \\ P : W :: d : C; \end{array}\right\} \qquad (132)$$

171. Friction affects the screw in the same manner that it does the wedge; the friction of the solid upon the hollow screw being sufficient to prevent it from returning after the power ceases to act. Hence the screw may be applied to many of the purposes for which the wedge is used. The difference in their use is, that in the wedge the power most frequently acts by a succession of blows; in the screw it acts in the manner of a constant force. Hence the latter may be used for raising great weights to a small height, and is the most frequent instrument used for accumulating force, in order to apply it to pressure. It therefore forms an essential part of the coining engine, the notaries' and printing, as well as various other presses. It is, also, like the modifications of the wedge, employed for fastening, and holding bodies together. This it does the more effectually, inasmuch as the screw must either be turned around and withdrawn, by reversing the motion by which it entered, or forced out, by breaking its own threads, or those of the hollow screw formed to receive it in the bodies to be united. It will therefore unite most bodies more effectually than nails or spikes, and is alone applicable to the union of such hard bodies as cannot be penetrated by any form of the wedge; such are, for instance, the metals, in the mass of which hollow screws, fitted to receive the solid screw, may be cut.

172. A screw may be applied to the teeth of a wheel, and will, by its simple revolution, cause the wheel to revolve. In this case, the screw need have no progressive motion, and is therefore formed upon a rod that is free to turn, but does not advance forwards. The hollow screw is no part of such a system, and the machine is a combination of the screw with the wheel axle.

In such a combination, the screw is called endless. It is represented beneath:



The power is applied to the head of the screw, the weight to the circumference of the axle: this power will be in equilibrio with a force W′, acting where the threads of the screw tend to turn the wheel, at whose circumference, by (132),

$$W' = \frac{PC}{d}, \text{ or } P = \frac{W'd}{c};$$

and the force W′ will be in equilibrio, with the weight acting upon the circumference of the axle, where, by (119 *a*),

$$W' = \frac{W'a}{b};$$

hence

$$P = \frac{Wdb}{ca}. \qquad (133)$$

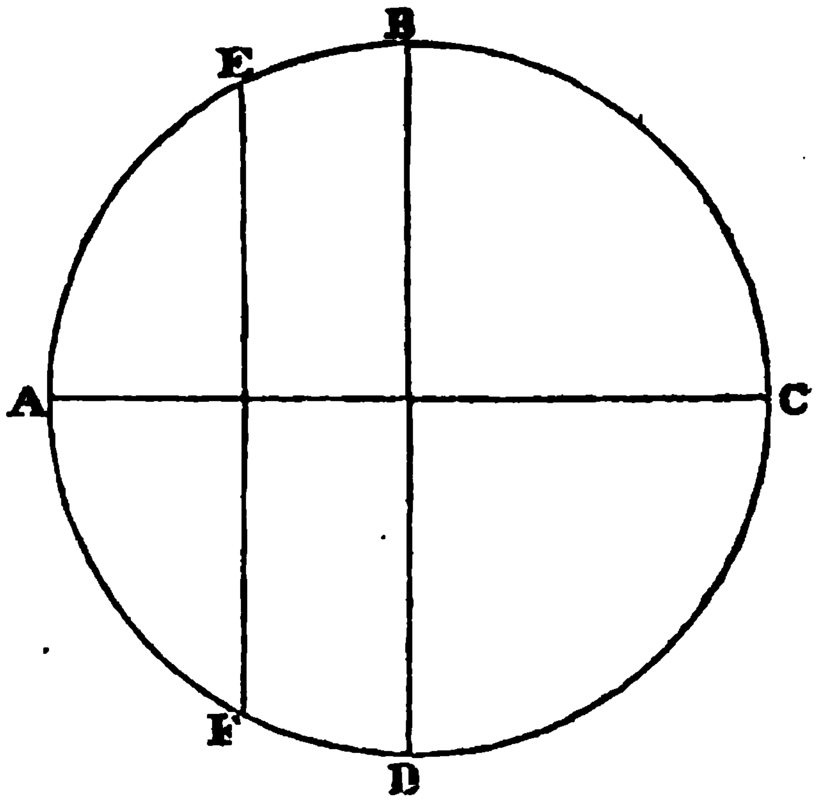
Equilibrium, therefore, will exist in the case of the perpetual screw acting upon a wheel and axle, when the power is to the weight as the distance between the threads of the screw multiplied by the radius of the axle, is to the circumference of the head of the screw multiplied by the radius of the wheel.

173. If the solid screw, and the hollow one to which it is adapted, be of a hard material, the motion cannot be free, unless the

intervals between their respective threads be as equal as the nature of materials will admit. As screws may be cut with many threads, within a small space, they may be applied to divide that space into as many parts as there are threads of the screw within it, and each of these parts will be determined by a complete revolution of the screw around its axis. But as the screw may have a circular head adapted to it, and the circumference of this head may be divided into many equal parts, the division may be carried further, to the determination of as many parts of the interval between the threads, as there are divisions upon the head of the screw; for as a complete revolution of the head of the screw corresponds to a progressive motion of the axis of the cylinder on which it is cut, through a distance equal to the interval between the adjacent threads; so a partial revolution will correspond to a progressive motion in the screw, that bears the same ratio to the distance between the threads, that this partial revolution bears to the whole circumference. In like manner, if a perpetual screw be applied to the teeth of a wheel, and both be of a hard material, the teeth, catching in succession in the thread of the same screw, must be equal among themselves. A complete revolution of the screw will move a point upon the circumference of the wheel, through a space equal to the distance between the threads, or to the breadth of the tooth upon the wheel; if a head be adapted to the screw, and its circumference divided into equal parts, portions of the revolution of the screw may be estimated by means of them, and these will be the measure of corresponding parts of the progressive motion of the wheel, through the interval between the threads of the screw.

This property of the screw is applied to several important purposes.

(1.) Many of the micrometers which are used in telescopes, for the measurement of such angles as are included within their field, are moved by means of screws; the numbers of whose revolutions, and the parts of a revolution, determined by the divisions of the head of the screw, furnish the measure of the angle. The whole breadth of the field is first determined by observing the time it takes an equatorial star to traverse the diameter, and this time is reduced to degrees, minutes, and seconds. To take an instance: let the circle ABCD represent the field of a telescope, in which are seen the horizontal and vertical wires AC, BD; the wire EF, is moved by the micrometer screw; the breadth of



the field having been determined by observation, it is next ascertained by experiment how many revolutions and parts of a revolution are required to move the wire across the field; and the value of each revolution and part is calculated by a simple proportion. The moveable wire is then placed so as to appear to coincide with the vertical one, and one of the objects being made to coincide with the latter, the screw is turned until the moveable wire coincides with the other object; then counting the revolutions of the screw, and observing the portions in excess upon the head of the screw, it will be at once seen that a measure of the angle is obtained in terms of the known divisions of the screw.

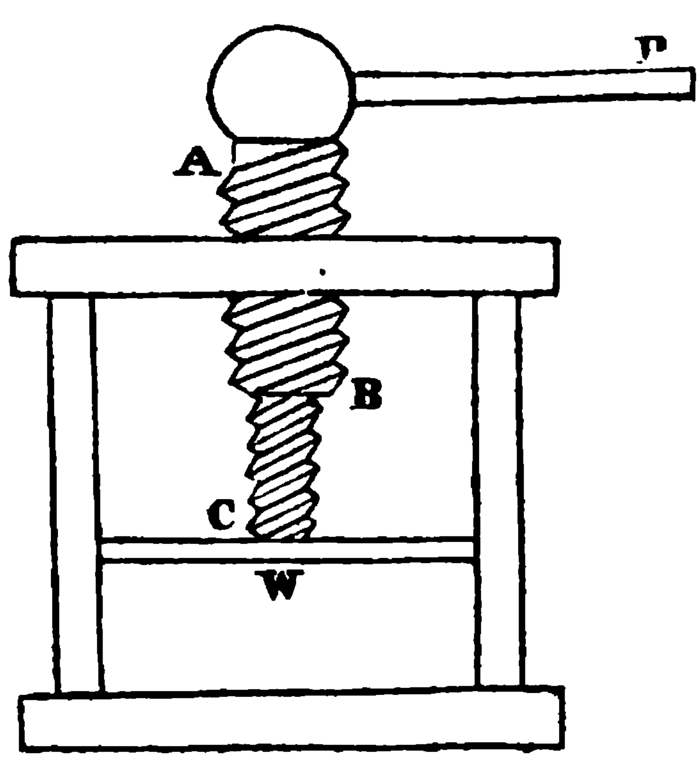
(2.) A screw moving a straight bar forward, affords the means of dividing it equally, and to great minuteness, by means of divisions on the head of the screw. Such is the principle of the Straight-line Dividing-Engine of Ramsden.

(3.) A circular plate may be made to revolve by means of an endless screw. A circular limb laid upon the plate, and concentric with it, may be divided, by ascertaining the proportion each thread of the screw bears to the whole circumference of the circle. More minute divisions may be obtained by dividing the head of the screw. Such is the principle of the circular-dividing engine of Ramsden. In a very beautiful application of the same principle, which is now used by Mr. Patten, an instrument maker, in New-York, the circular plate has 360 teeth on its circumference: each revolution of the screw, therefore, corresponds to a single degree, and the more minute divisions are obtained by divisions upon its head. For divisions of instruments that do not require much care, the screw is turned by a treadle, at each motion of which, a cutting tool makes its mark upon the

limb to be divided. The same engine has been applied to straight lines, by placing upon the circular plate, a second plate accurately centered, whose circumference is exactly a yard; the rod to be divided is moved forward by friction against this plate, and consequently, by a complete revolution of the plate, through a straight line a yard in length: ten revolutions of the screw, therefore, correspond to an inch, each revolution to the tenth of an inch, and lesser divisions are obtained by means of the divisions upon the head of the screw.

(4.) The same principle was applied in the astronomical instruments of the last century, to subdivide the smallest divisions cut upon their limbs. As this method has now been superseded by others, it requires no explanation in the present treatise.

174. It will appear from reference to § 172, that the increase in the intensity of the power, will, in the screw, if the diameter of its head, or the length of the bar applied to turn it, remain constant, be in the inverse ratio of the distance between the threads; hence, in order that a small power shall counterbalance a weight of great intensity, the thread must be very much diminished. In this event, the strength of the thread may be so far lessened as to be incapable of bearing, without breaking, the intensity of the resistance. A common screw, then, has a limit to its capacity of changing the intensity of the force, in the strength of the material of which it is constructed. Screws, with fine threads, have the additional disadvantage, that, after their work is performed, it requires a considerable time to turn them back again to the point at which their action commenced, in order to apply them to overcome a new resistance. Both of these defects have been remedied by a compound screw, invented by Dr. Hunter. In the figure, AB is a screw of large thread, and consequently of



little power; but it is hollow within, and cut into threads, to receive the second solid screw, BC. The threads of the latter are a little less than those of the former. Were they exactly equal, when the screw AB is turned, the screw BC would enter into its cavity just as far as the former is pressed forward, and the point C would remain at rest. But as the threads of BC are smaller than those of AB, the former recedes during a single revolution of the latter, through a space equal only to its own thread; and hence the point C is pressed forward, through the difference between the respective threads. The velocity of the weight then is no more than the difference in the velocities of the screws, and by the principle of virtual velocities, which is most convenient for our purpose, the condition of equilibrium is: the power shall be to the weight, as the circumference of the circle described by the power, is to the difference between the distances of the threads in the two screws. If now there be ten threads of the outer screw in the space of an inch, and ten of the inner in the space of nine-tenths of an inch, the difference will be $\frac{1}{100}$th of an inch, and the intensity of the power is as much increased as it would be in acting upon a screw having an hundred threads within an inch; while the strength of the threads of the least screw would be more than nine times as great, and capable, therefore, of bearing more than nine times the strain. It is easy in practice to make the greater screw carry the lesser back with it to its primitive position, and hence it can be placed in the position in which it is again to begin to act, in a tenth part of the time, in the case we have assumed, and in proportion for any other difference between the distances of the threads. A third screw may be placed within the second; and the intensity of the power again increased upon the same principle.

175. The friction of a simple screw depends not merely upon the nature of the materials, and their polish, but also upon the greater or less tightness with which the solid and hollow screws apply themselves to each other. It would in consequence be impossible to reduce the action of the friction to any fixed law. In the endless screw, when combined with the wheel and axle, it may be investigated, and the equation of the state in which an addition to the power, however small, would cause motion, will be found to be, using the notation of (133)

$$P=W.\ \frac{b}{a}\cdot\frac{d+fc}{2C-fd}. \qquad (134)$$

CHAPTER VII.

Of the Strength of Materials.

176. The force with which the particles of a solid body resist the forces that tend to separate them, and which constitutes their strength, may be found, in particular cases, by experiment. If the experiments be multiplied, both in respect to the species of substances, and to the size and circumstances under which portions of a given substance are employed, a general law might be finally obtained, whence algebraic expressions of the relation between the strength, the dimensions, and the material, might be deduced. Or we may, by assuming an hypothesis to represent the probable manner in which bodies are made up, deduce from it general formulæ, which, applied to cases that occur in practice, will be sufficient, in almost every instance, to represent the phenomena. The latter of these methods, although not the most accurate, shall be employed by us; but we shall carefully note the differences that exist between the inferences from the hypothesis, and the results of experiments.

177. Galileo was the author of the hypothesis that is most generally employed by writers on mechanics, and which will suffice in all usual cases. He assumes that all solid bodies are composed of a great number of parallel and equal fibres, perfectly inflexible and inextensible; that when they break, the several fibres give way in succession, and the body turns upon the fibre or fibres that are the last to give way, as upon a hinge.

Leibnitz observing the flexure that takes place in bodies before they break, assumed as the basis of his hypothesis, the fact, that every body admits before it breaks, of a certain degree of extension: the fibres, therefore, are both extensible and flexible; and he inferred that the strength of each fibre, instead of being equal, varied with its quantity of extension, or was proportioned to its distance from the fixed point around which the beam is supposed to turn.

The hypothesis of Galileo has been recently extended by Barlow, who has, by introducing the circumstance of the occurrence of a flexure before the beam breaks, been enabled to explain all the cases which appeared, by a comparison of the hypothesis with experiment, to be anomalous.

178. The force which tends to destroy the aggregation of the particles of a solid body, may act in four different manners.

(1.) It may tend to tear the body asunder by exerting an action in the direction of its fibres.

(2.) It may tend to break the body across, by a transverse strain, exerted either perpendicularly, or obliquely to the direction of its fibres.

(3.) It may tend to crush the body.

(4.) It may tend to separate the particles by means of torsion, twisting or wrenching the body, by an action in a plane perpendicular to its axis.

The resistance which bodies oppose to a force acting in the first of these modes, is called the Absolute Strength. In fibrous bodies, it is different, according as the force is applied in the direction of the fibre, or at right angles to it: for although by hypothesis we should conceive fibres to exist, even in the transverse direction, yet, in nature, none such are found; and the aggregation is far more easily destroyed in the latter case than in the former.

The resistance to a fracture across the body, is called its Respective Strength; and we may call the resistance to a twisting force, the Strength of Torsion.

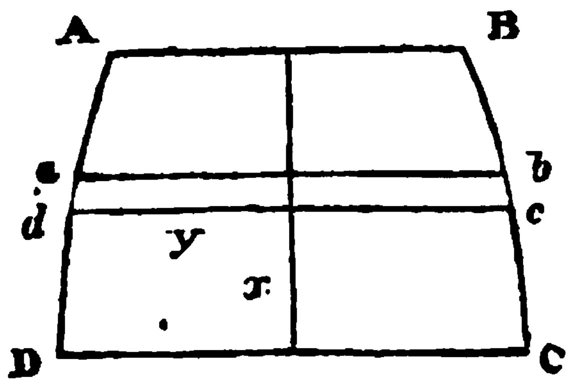
It needs no reasoning to show that the measure of the strength, even in the same body, will be different in each of these different cases; and such is the difference in the manner in which the particles of bodies of different natures are united, that there can be no general law that will represent the relations of these four species of resistance. But in conformity with our hypothesis, it will be obvious that, in all the several cases, the resistance to fracture will vary with the number of fibres, which, in homogeneous bodies will depend upon the area of their sections. It must also depend upon the manner in which the force acts to break the body.

Of the Absolute Strength of Materials.

179. When the strain is exerted in the direction of the fibres, the force that tends to break a body will be directly opposed to its force of aggregation, and the resistance must depend upon the cohesive force of each fibre, and upon their number, but upon no other circumstance.

To enable us to express this analytically:

Let ABCD, represent the area of a prismatic beam or rod; x, and y, the ordinate and abscissa of any point of its surface; s the absolute strength of one of its differential elements, $a\ b\ c\ d$. The force of the aggregation, may be represented by the weight which is just sufficient to destroy it, which we shall call U.



The area of the section will be

$$2\int y dx.$$

The expression for the absolute strength will therefore be

$$U=2s\int y dx, \tag{135}$$

for the area of each of the elements will be $2ydx$, its strength $2sydx$, and the strength of the whole will be found by integrating this expression, in which $2s$ is constant.

In a rectangular bar, whose lineal dimensions are a and b

$$ab=2\int y dx,$$

and

$$U=sab, \tag{136}$$

whence

$$s=\frac{ab}{U} \tag{137}$$

In a circle whose radius is r,

$$2\int y dx=\pi r^2,$$

and

$$U=\pi r^2 s, \tag{138}$$

whence

$$s=\frac{U}{\pi r^2}. \tag{139}$$

180. When experiments are made upon the resistance of rods, of dimensions given in some conventional unit of lineal measure, to a direct strain, exerted by means of loads estimated in some conventional unit of weight; the value of s may be found in terms of the latter, and in reference to an element of the surface, whose magnitude is the square of the unit of length. The most valuable experiments that we have of this sort, are those of Barlow, upon wood. These were made upon cylindrical rods, and the strength, s of the formulæ, deduced for an element of the area of a square inch. The results are contained in the second of the following tables. The first table has been compiled from various other sources.

TABLE I.

ABSOLUTE STRENGTH OF THE METALS.

Cast Steel, - - - - -	140000 lbs.
Gold, (*according to Morveau*)	80000 lbs.
Wrought Iron, (Swedish,) -	72000 lbs.
do (English,) -	56000 lbs.
do do in the form of chains,	48000 lbs.
Bronze, (Gun Metal,) - -	36000 lbs.
Wrought Copper, - - -	33000 lbs.
Cast do - - -	19000 lbs.
Brass, - - - - -	17000 lbs.
Tin, - - - - -	4700 lbs.
Lead, - - - - -	1800 lbs.

TABLE II.

ABSOLUTE STRENGTH OF DIFFERENT KINDS OF WOOD DRAWN IN THE DIRECTION OF THEIR FIBRES.

Boxwood, - - - -	20000 lbs.
Ash, - - - - -	17000 lbs.
Teak, - - - - -	15000 lbs.
Norway Fir, - - - -	12000 lbs.
Beech, - - - -	11500 lbs.
Canada Oak, - - -	11400 lbs.
Russia Fir, - - - -	10700 lbs.
Pitch Pine, - - - -	10400 lbs.
English Oak, - - -	10000 lbs.
American White Pine, - -	9900 lbs.
Pear Tree, - - - -	9800 lbs.
Mahogany, - - - -	8000 lbs.
Elm, - - - - -	5800 lbs.

TABLE III.

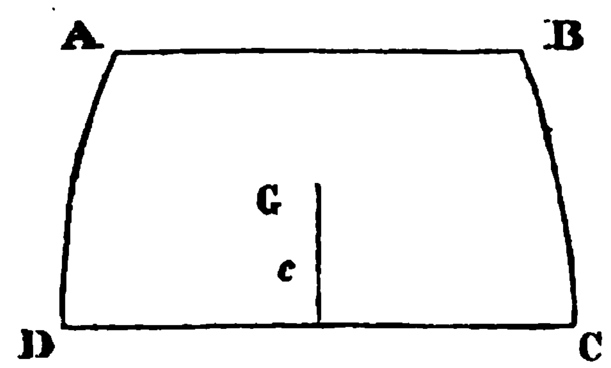
ABSOLUTE COHESIVE STRENGTH OF WOOD DRAWN IN A DIRECTION AT RIGHT ANGLES TO THE FIBRES.

Teak, - - - - - -	818 lbs.
American White Pine, - -	757 lbs.
Norway Fir, - - - -	648 lbs.
Beech, - - - - -	615 lbs.
English Oak, - - -	598 lbs.
Canada Oak, - - -	588 lbs.
Pitch Pine, - - - -	588 lbs.
Elm, - - - - -	509 lbs.
Ash, - - - - -	359 lbs.

Of the Respective Strength of Materials.

181. To apply the hypothesis of Galileo to the case of a transverse strain, we shall suppose in the first place, the substance to have the form of a prismatic beam, that it is firmly inserted at one end into a fixed support, lies in a horizontal position, and is acted upon by a weight that presses at the end that is not fixed; that the fracture takes place at the point of support, beginning at the upper side, on which the weight presses, and terminating at the other. The beam then, its fibres being by hypothesis inflexible and inextensible, will turn around the latter point in a vertical plane. At the instant of fracture, the two forces that act are in equilibrio with each other, their respective moments of rotation must be therefore equal.

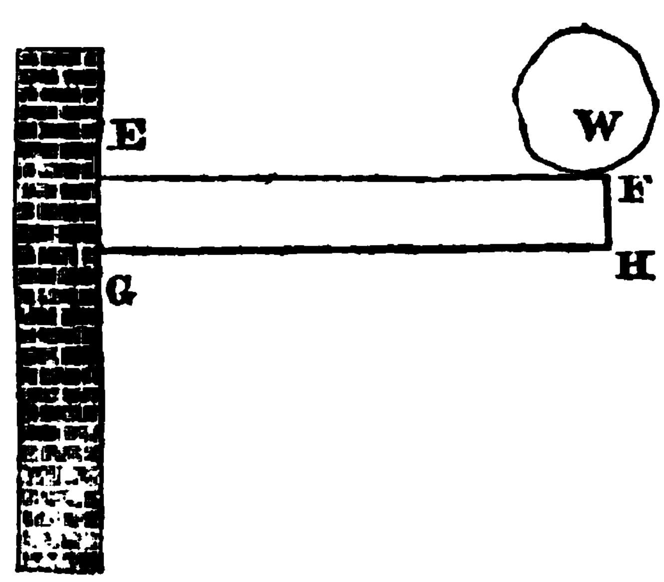
In the prismatic beam, whose section is ABCD, the strength will be represented by the expression (§ 135), $2s\,ydx$; but as the effort of the weight will cause the beam to turn around a horizontal axis passing through the lowest point, the moment of rotation of the strength of each element, will be found by multiplying $2s\int ydx$ into the perpendicular distance of its centre of gravity from this axis.



Now the mean of all these distances will be the distance of the centre of gravity G, of the whole, from the point where the fracture terminates. The moment of rotation of the whole resistance will therefore be, calling this distance c,

$$2cs\int ydx.$$

Let EFHG represent the longitudinal section of the beam,



fixed at EG to a firm support, and pressed by a weight acting at F. Let the length EF$=l$; the moment of rotation of the weight will be lW, and at the instant of breaking,

$$\left.\begin{aligned} lW &= 2cs\int ydx \\ \text{or}\qquad W &= \frac{2cs\int ydx}{l} \end{aligned}\right\} \qquad (140)$$

In rectangular beams, whose two dimensions are a and b,

$$2\int ydx = ab,$$

$$c = \frac{b}{2},$$

and the equation (140) becomes

$$W = \frac{sab^2}{2l}. \tag{141}$$

In square beams, whose side is a,

$$W = \frac{sa^3}{2l}. \tag{141 b}$$

In cylindric beams, whose radius is r,

$$W = \frac{\pi r^3 s}{2l}. \tag{142}$$

In a hollow cylindric beam, whose cavity is a cylinder having the same axis as the outer surface, the radii of the two cylinders being R and r',

$$W = \frac{(\pi R^2 - \pi r'^2) Rs}{l}; \tag{143}$$

If the area of the cylindric ring be equal to πr^2 of (142)

$$W = \frac{\pi r^2 Rs}{l}; \tag{144}$$

and the strength will be to the strength of the solid cylinder of (142) as R : r, that is to say: In different cylindric beams, having the same quantity of the same material, and equal lengths, but having different diameters, in consequence of cavities of greater or less size within them, the strengths are in the direct ratio of their diameters.

We may obtain the comparative dimensions of solid and hollow cylinders, that will bear equal weights, by a comparison of the formulæ (142) and (143), whence we obtain

$$\left.\begin{array}{l} r^3 = R^3 - Rr'^2 \\ r = \sqrt[3]{[R(R^2 - r'^2)]}. \end{array}\right\} \tag{145}$$

If the cavity of the hollow cylinder be not concentric, the strength should increase, according to the hypothesis, as the cavity approaches the lower side, when it is fixed at one end only; for in this case, the centre of gravity of the section will be further removed from the point where the fracture terminates.

In a beam of the shape of an isoceles triangle, whose base is e, and altitude i,

$$2\int ydx = \frac{ei}{2};$$

if the edge be placed uppermost,

$$c = \frac{i}{3},$$

and

$$W=\frac{ei^2s}{6l};\tag{146}$$

if the base be placed uppermost,

$$c=\frac{2i}{3l};$$

and

$$W=\frac{ei^2s}{3},\tag{147}$$

In a square beam, whose diagonal is vertical,

$$2\int ydx=a^2,$$

$$c=\frac{a}{2}\sqrt{2},$$

and

$$W=\frac{sa^3\sqrt{2}}{2l}.\tag{148}$$

In a rectangular beam, when l and s are constant, the strength varies with ab^2, (141). This proposition may be applied to a case that may sometimes be useful in practice, namely: To cut the strongest possible beam out of a given cylinder. Thus a tree, although a cone or conoid, may, for all useful purposes, be considered as a cylinder; for the size of the rectangular beam that can be cut from it, will be determined by the area of its smaller end.

Let r be the given diameter of the cylinder, x and y the lineal dimensions of the required beam. When this is the strongest possible, xy^2 will be a maximum;

$$y^2=r^2-x^2;$$

$$xy^2=r^2x-x^3;$$

$$r^2dx-3x^2dx=0;$$

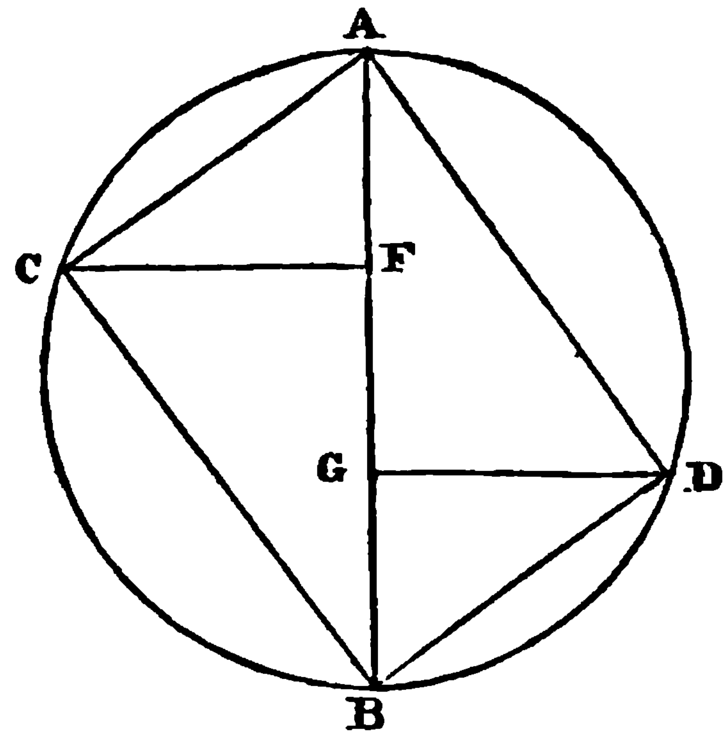
$$r^2=3x^2;$$

$$y^2=2x^2;$$

$$x^2:y^2:r^2::1:2:3.\tag{149}$$

The cylinder must therefore be so cut, that the squares of the breadth of the beam, of its depth, and of the diameter of the cylinder, shall be in the proportion of 1 : 2 : 3.

This admits of an easy geometric construction. In the circu-



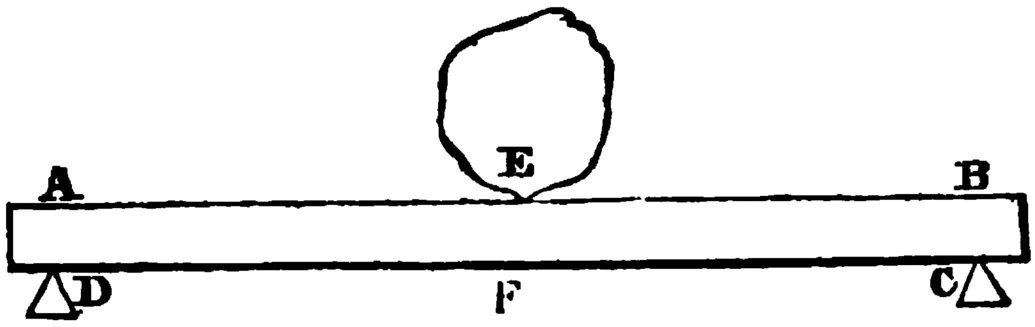
lar section of the cylinder draw the diameter AB, divide the diameter into three equal parts by the points F and G; from the points F and G, draw the perpendiculars FC, GD, towards opposite sides, cutting the circle in the points C and D; join AC, CB, BD, AD; the parallelogram ADBC, will have the required property, and will be the section of the strongest beam that can be cut from the cylinder, whose diameter is AB.

It will be at once seen that

$$AB^2 : BC^2 : AC^2 :: 1 : 2 : 3. \qquad (150)$$

182. When a beam, lying in a horizontal position, rests upon two props, and is broken by a weight placed at an equal distance from the two props, we may consider the laws of its strength as included in the general case of a beam supported at one end only; for if, according to the hypothesis, it break without bending, we may conceive it to be formed of two beams, each inserted in a firm support at the place of fracture, and acted upon at each end by a force equal to half the weight that just breaks it, but which is directed upwards instead of downwards. This force, which is equal to half the weight, will act at a distance which is equal to half the length; its effort is therefore equal to no more than a fourth part of the effort of the same weight, applied to the same beam, if supported at one end only; and as this effort must be just equal, at the instant of breaking, to the transverse strength of the beam, the latter will be four times as strong as when supported at one end only.

In the beam ABCD, supported at C and D, and to which a



weight, W, is applied at the point E, which bisects AB. If we suppose the half AEFD, to be firmly fixed at EF, it will be broken by a force applied at A, in the vertical direction, which is equal to $\frac{1}{2}$ W; the moment of rotation of this force will be equal to the respective strength of the beam, and as the distance EA is $\frac{1}{2}l$, (140)

$$\frac{Wl}{4}=2sc\int ydx,$$

and

$$W=\frac{8sc\int ydx}{l}; \tag{151}$$

In rectangular beams, one of whose faces is horizontal, we obtain in the same manner as (141),

$$W=\frac{2sab^2}{l}. \tag{152}$$

In square beams, (141 b),

$$W=\frac{2sa^3}{l}. \tag{153}$$

In cylindric beams, (142),

$$W=\frac{4\pi r^3 s}{l}; \tag{154}$$

with a similar inference for the case of a concentric hollow cylinder, as in (143) and (144).

But if the hollow be not concentric, the strength will, in the present case, increase with the approach of the cavity to the upper surface. And so in triangular beams, the proportions of (146) and (147) will still hold good, but they will be stronger with the edge uppermost.

In a square beam, whose diagonal is vertical, (148),

$$W=\frac{2sa^3\sqrt{2}}{l}. \tag{155}$$

183. In the case of a beam lying horizontally, and firmly fixed at each end, the resistance will be equal to that of a single beam supported by two props, added to those of two beams fixed at one end only; for it is obvious, that three fractures must be produced, one in the middle, and one at each end; at the first, the re-

sistance will be equal to that of the supported beam, or four times as great as that of the same beam, if supported at one end only. The resistance in each of the latter cases, will be that of a beam of half the length fixed at one end only, or one fourth of the last resistance; the whole resistance will therefore be six times as great as that of the same beam when fixed at one end only, or

$$W = \frac{12sc \int y dx}{l}; \tag{156}$$

In rectangular beams, (141) and (152),

$$W = \frac{3sab^2}{l}. \tag{157}$$

In square beams, (141 *b*) and (153),

$$W = \frac{3sa^3}{l}. \tag{158}$$

In cylindric beams, (142) and (154),

$$W = \frac{6\pi r^3 s}{l}. \tag{159}$$

If experiments be made with rectangular beams, in either of the three several positions, the absolute strength, s, is determinable by means of them, for calling the weights that just break them, W, W′, and W″.

In beams firmly fastened at one end, from (141),

$$s = \frac{2lW}{ab^2}; \tag{160}$$

In beams, supported at each end on props, from (152)

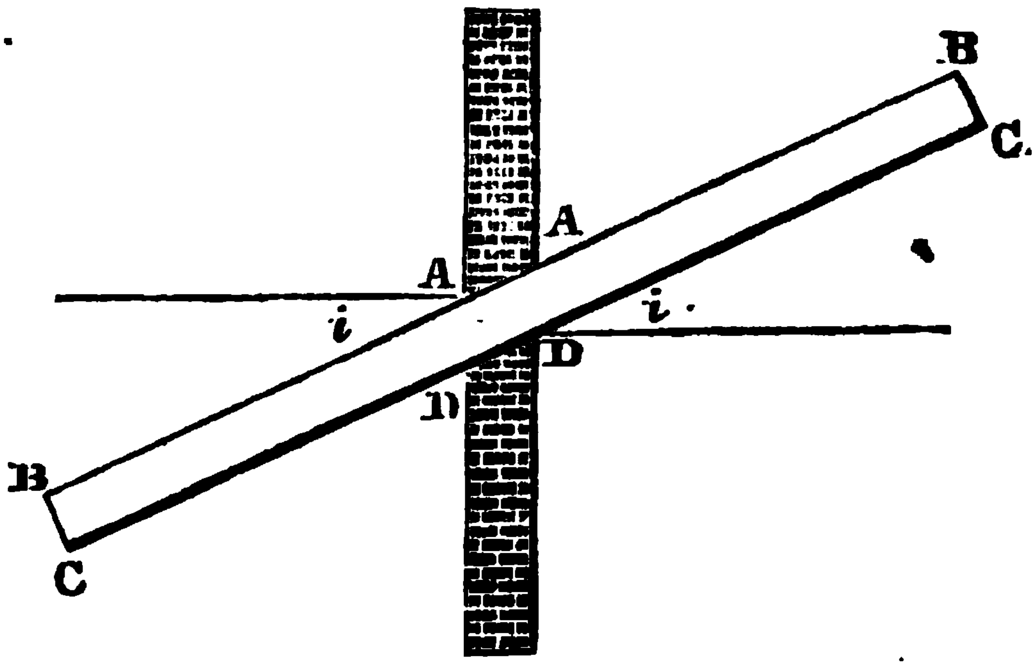
$$s = \frac{lW'}{2ab^3}; \tag{161}$$

In beams firmly fastened at each end, from (157)

$$s = \frac{lW''}{3ab^2}; \tag{162}$$

184. When the beam, instead of lying in a horizontal position, is inclined, an increase of its strength takes place, which we shall proceed to investigate.

Let either of the beams, whose longitudinal section is ABCD,



lie in an inclined position, and let the angle of inclination be i, the moment of rotation of the weight will become

$$Wl \cos. i. \tag{163}$$

The effort the weight exerts to break the beam, is no longer exerted directly, but it is unnecessary to take this obliquity into account, if bodies be constituted, as represented by our hypothesis; for the number of fibres that act to resist fracture, is still the same. In bodies that are not fibrous, as, for instance, in those that are crystallized, the inference would probably be different; but, on this point, we are unable to refer to the results of any experiments. It will be obvious that the same reasoning will be true, whether the beam be inclined upwards or downwards, and is applicable to the cases of its being supported, or firmly fixed at both ends, as well as to that of its being fixed at one end only.

185. When the weight that tends to break a beam is not accumulated at a single point, but is uniformly distributed over its whole length; its effort is diminished to the half of what it exerts, when, in the case of a beam fixed at one end, it acts at the opposite extremity; or when, in the case of a beam supported or fixed at both ends, it acts in the middle. This will be obvious in the first case, from the consideration that the point, where the weight acts, will be in the middle of the beam, instead of being at its end; hence its moment of rotation becomes $\frac{1}{2}\ lW$; and as the other two cases are deduced immediately from the first, the same principle applies to them also.

186. We have hitherto omitted the action of the weight of the beam itself.

In small beams indeed, their own weights are of little importance, and need hardly be taken into account in the experiments; but in large beams this is not the case, as may easily be seen.

The weight which breaks a beam is made up of its own weight, and the weight which is applied for the purpose: the former acts at the centre of gravity of the beam, which in prismatic beams is in the middle of their length. Its moment of rotation, therefore, will be $\frac{1}{2}$ Vl, calling the weight of the beam V; the joint effort of the two will therefore be, in the case of a beam supported at one end, and placed horizontally, and if the additional weight act at the extremity,

$$(W+\tfrac{1}{2}V)l;$$

and the formula (140) for the strength will become

$$W+\tfrac{1}{2}V=\frac{2sc\int ydx}{l}. \tag{164}$$

In beams that are similar, we may substitute, for $2c\int ydx$, the cube of any one of their homologous dimensions multiplied by a constant co-efficient; let then

$$fl^3=2c\int ydx:$$

and the above equation becomes

$$W+\tfrac{1}{2}V=sfl^2. \tag{165}$$

In similar beams of the same homogeneous material, the weight is a function of the cube of their homologous dimension, as will be half the weight, or

$$\varphi l^3=\tfrac{1}{2}V,$$

and

$$W+\varphi l^3=sfl^2. \tag{166}$$

It will therefore be evident, that while the strength of similar beams increases only as the square of one of their homologous dimensions, the effort of their own weight to break them increases with the cube; and thus a limit will be reached, when

$$\left.\begin{array}{l}\varphi l^3=sfl^2\\ \text{and}\quad W=0.\end{array}\right\} \tag{167}$$

The same principle applies equally to the other two cases, in which beams are supported, or fixed at both ends.

187. We shall now recapitulate the results of our hypothesis, and state what discrepancies have been observed between them, and the inferences from actual experiment.

(1.) In any prismatic beam whatsoever, the strength is directly proportioned, to the area of its section, and to the distance of its centre of gravity from the point where the fracture terminates; and inversely, to the length of the beam.

(2.) The strengths of beams lying in a horizontal position, when fixed at one end only; when supported by a prop at each end; and when firmly fixed at both ends, are as the numbers 1 : 4 : 6. That is to say: a beam firmly fixed at both ends, is six times, a beam merely supported at both ends, four times as strong as when it is fixed at one end only.

These several inferences from the hypothesis, agree within all usual limits, with the results of experiments. The discrepancies are: that the strengths increase in a ratio a little greater than the square of the depth, in rectangular beams; and decrease rather more rapidly than the inverse ratio of the length.

The second of the above propositions admits of the following cases:

(*a*) In beams of the same material, with equal and similar sections, and unequal lengths, the strengths are inversely proportioned to the lengths.

The lengths being equal:

(*b*) In rectangular beams of the same materials, the strengths are proportioned to the product of the breadth by the square of the depth;

(*c*) In square beams, the strengths are proportioned to the cubes of the sides of the square sections;

(*d*) In solid cylindric beams, the strengths are proportioned to the cubes of the radii;

(*e*) In hollow cylindric beams, having the same quantities of material distributed around cylindric cavities of different diameters, the strengths are directly as the diameters.

(3.) Large beams are weaker in proportion than small ones, for their own weight constitutes a part of the force that tends to break them; and in similar solid bodies, the stress growing out of their own weight increases as the cubes of their homologous dimensions, while the strength only increases with the squares.

We see from this, that models may be strong, and capable of bearing a stress far beyond any that can be applied to them; yet that machines constructed exactly similar to them in proportion, and of like materials, but of increased dimensions, may become too weak to bear even their own weight; that there must be a limit to the size and extent of any structures that can be erected by the hand of man; and that a similar limit exists even in the works of nature. Thus in organic bodies, mountains, hills, trees, the size they can attain, without risk of disintegration, is restricted within certain bounds. In the animal creation, the same principle applies, and the limit is sooner reached.

Our theory would show that when a body becomes weak in consequence of an increase of length, strength may at first be added by increasing its breadth, and still more by increasing its thickness, the length remaining constant. Here, however, the weight is increased in a greater ratio than the length, and finally becomes excessive. The same quantity of material may assume a stronger form by being fashioned into a hollow tube; yet here again a limit is reached when, the circumference of the

tube becomes so thin as to be liable to be crushed by the forces that act upon it.

In animals, we find that the smaller classes have bones far more slender in proportion than those of the larger kinds. Their muscles are far less thick in proportion to their length; and their masses are diminished in a proportion much more rapid than that of the cubes of their similar dimensions. We find small animals capable of lifting weights greater in proportion to those of their own bodies, than larger animals can; and in spite of this additional effort, they are enabled to continue their exertions for a longer period without fatigue. To obtain the greatest possible strength with the least possible weight, the bones of animals have the form of hollow tubes, as have the quills and feathers of birds. In the vegetable kingdom, we find trees and plants made up of bundles of hollow tubes; and in those where great strength and comparative lightness are necessary, these are again arranged so as to form a hollow cylinder; as, for instance, in the whole family of the gramina.

(4.) When beams are in an inclined position, their strength, which we shall call F, becomes

$$F=W \cos. i. \qquad (168)$$

This deduction from the hypothesis, is true in practice, so long as the beam does not bend under the effort of the weight applied to break it.

(5.) When the pressure, instead of acting upon a single point, at the extremity of a beam fixed at one end, or in the middle of a beam supported or fixed at both ends, is equally distributed throughout the whole beam, twice the weight will be required to break it.

188. As far as we have recited the results of the hypothesis, they agree in all useful cases with the deductions from experiment. But some of the rules, deduced from the hypothesis, do not coincide with what occurs in practice.

Thus:

It has been shown that there is a difference in the strength of triangular beams, according to their position, with an edge or a face uppermost; and that this difference follows different laws, according to the manner in which the beam is supported. Experiment absolutely contradicts this; for in neither of the three different positions in which beams have been tried, has any important difference been found in the strength of a triangular beam, when placed with an edge, or with a face uppermost.

So also, hollow cylindric beams have not been found stronger, when the cavity has been nearer to the side where the fracture terminates, as they ought to be in conformity with hypothesis; neither is a square beam stronger when its diagonal is vertical.

These remarkable discrepancies, together with the less important ones that have been noted in the preceding section, arise from an omission in the hypothesis; this does not take into account the elasticity and consequent flexure which materials undergo before they actually break. These circumstances have been introduced into the investigation by Barlow, in his treatise on "the Strength and Stress of Timber;" to this work, then, we refer for the mode in which the following formulæ, that coincide almost exactly with the results of experiment, have been obtained. Using the same notation as before, and calling the angle of deflection d,

In a beam fixed at one end and loaded at the other,

$$s' = \frac{l \mathrm{W} \cos. d}{ab^2}; \tag{169}$$

In a beam supported at both ends, and loaded at the middle,

$$s' = \frac{l \mathrm{W}}{4ab^2 \cos.^2 d}; \tag{170}$$

In a beam fixed at each end, and loaded in the middle,

$$s' = \frac{l \mathrm{W}}{6ab^2 \cos.^2 d}. \tag{171}$$

In these formulæ, s' differs from s in our formulæ (160), (161), and (162), inasmuch as it is the strength of the element ydx in (140), (151), and (156), instead of being the strength of $2ydx$. We have preferred our mode of estimating the respective strength, inasmuch as it retains the connexion with the formula (139), which gives the value of the absolute strength.

189. In order to make our formulæ applicable to practice, we subjoin a table of the respective strength of various bodies, reduced to an element of the size of a cubic inch.

TABLE

OF THE RESPECTIVE STRENGTH OF VARIOUS SUBSTANCES.

	Metals.	
Wrought Iron,	Swedish,	22000 lbs.
do	English,	18000 lbs.
Cast Iron,		16000 lbs.
	Wood.	
Teak,		4900 lbs.
Ash,		4050 lbs.
Canada Oak,		3500 lbs.
English Oak,		3350 lbs.
Pitch Pine,		3250 lbs.

Beech, - - - - -	3100 lbs.
Norway Fir, - - -	2950 lbs.
American White Pine, - -	2200 lbs.
Elm, - - - - -	1013 lbs.

We are without any good experiments on the respective strength of stone. It would, however, appear from some experiments of Gauthey, that, in soft freestone,

$$s=68.7 \text{ lbs.}$$

In hard freestone,

$$s=72.75 \text{ lbs.}$$

And from some experiments recorded by Barlow, that in brick

$$s=64 \text{ lbs.}$$

This would make the respective strength of stone and brick, far beneath that of wood, or iron.

The preceding table may be applied to the calculation of the strength of horizontal beams, of any figure whatever, by the three formulæ, (140), (151), and (156). These may, however, assume a more convenient form for practice, by calling the area of the beam's section, A.

Let then—

$A=2\int ydx$, the area of the transverse section of the beam in sq. inches;

c=the distance of the centre of gravity of A, from the point where the fracture terminates, in inches,

l=the length of the beam in inches,

s=the number from the preceding table.

W=the measure of the beam's strength, being the weight which will just break it in pounds.

In beams fixed at one end, and loaded at the other,

$$W=\frac{csA}{l}. \qquad (172)$$

In beams supported at each end, and loaded in the middle,

$$W=\frac{4\,csA}{l}. \qquad (173)$$

In beams fixed at each end, and loaded in the middle,

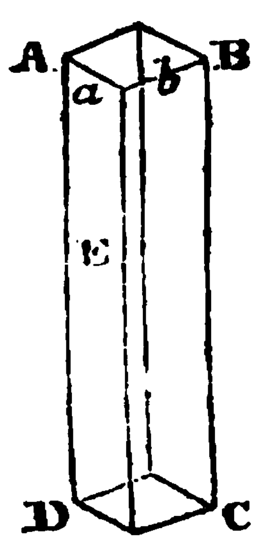
$$W=\frac{6\,csA}{l}. \qquad (174)$$

The particular formulæ, for rectangular square and cylindric beams, will be found at (141), (141b), (142), (152), (153), (154), (157), (158), and (159).

Of the Resistance of Bodies to a Force exerted to crush them.

190. The resistance of bodies to forces that act to crush them, would, at first sight, appear to follow the same law with the absolute strength; that is to say: it must be proportioned to the area of the body, and the force of aggregation of its particles. Experiment, however, shows, that the thickness of the substance has an important influence on the pressure it is capable of bearing, without being crushed. In the first place, it is found that very thin plates are readily crushed; their resistance next increases with the thickness, until it reach a maximum; and finally, decreases slowly, with a farther increase of thickness. It has been attempted to frame a mathematical theory, that should represent these circumstances; assuming, that the body was composed of flexible fibres, and that the crushing took place in consequence of a bending in the fibres.

The respective strength, F, of the pillar, ABCD, supposed to be rectangular, is represented by the formula (141).



$$F=\frac{sab^2}{2l},$$

the effort of the weight W, exerted to break it, may be represented by a force, bearing a constant relation to the weight and applied to the middle of the column, say, at the point E; its moment of rotation will be $\frac{Wl}{2}$, and,

$$\frac{Wl}{2}=\varphi\left(\frac{sab^2}{2l}\right);$$

whence,

$$W=\varphi\left(\frac{sab^2}{l^2}\right); \qquad (173)$$

and in a square beam,

$$W=\varphi\left(\frac{sa^3}{l^2}\right); \qquad (174)$$

191. In bodies whose sections are similar, it may be inferred that the resistance to a force exerted to crush them, is proportioned, directly to the cubes of the homologous dimensions of the sections, and inversely to the squares of their lengths. The most complete set of experiments that we have upon the variation in the strength of columns of different lengths, are those of Muschenbrook, and they correspond closely with the above theory. The subject, however, has not been so fully tested as to authorize us to receive any theory with implicit confidence. We shall, in

consequence, give the absolute results of experiments, in the terms of the weights and dimensions of the original record, without attempting to reduce them to a conventional unit.

TABLE.

EXPERIMENTS MADE BY RONDELET, ON THE RESISTANCE OF DIFFERENT SPECIES OF STONE IN CUBIC BLOCKS, OF THE SIZE OF 5 CENTIMETRES IN EACH DIMENSION.

	Spec. Grav.	Crushing Weight. *Kilogrammes.*
Swedish Basalt, - - -	3.065	47.809
Basalt of Auvergne, No. 1,	3.014	44.250
do No. 2,	2.884	51.945
do No. 3,	2.756	28.858
Porphyry, - - - -	2.798	50.021
Green Granite, (Vosges,) No. 1,	2.854	15.487
Grey do (Brittany,)	2.737	16.353
Granite, (Vosges,) No. 2,	2.664	20.482
Granite, (Normandy,) - -	2.662	17.555
Granite, (Champ du Boul,) -	2.643	20.441
Granite, (Oriental Rose,) -	2.662	22.004
Black Marble, (Flanders,) -	2.721	19.719
White Veined Marble, - -	2.701	7.455
White Statuary Marble, -	2.695	8.176

Experiments made at the same time upon cubes of stone, whose sides varied from 3 to 6 centimetres, showed that the strengths varied nearly in proportion to the areas of their bases, and were not influenced by the thickness. This corresponds to the law of absolute strength in § 179, and differs from that we have given for the resistance to crushing. On the other hand, when cubes of the same size were placed one upon another, a diminution in the resistance was found that does not differ much from the latter law.

By experiments made by Rennie, on the resistance of cast iron to pressure, it was found that the maximum strength was reached at thicknesses of from $\frac{3}{8}$ths to $\frac{1}{2}$ an inch, and that a prism a quarter of an inch square, and half an inch in depth, was crushed by 10,000 lbs.

A cube of cast copper, $\frac{1}{4}$ inch each way, was crushed by 7318 lbs.

A cube of wrought copper, of the same size, by 6440 lbs.

Of brass, by 10304 lbs.

Of cast tin, by 966 lbs.

Of lead, by 483 lbs.

An inch cube of elm, is crushed by 1284 lbs.

Of American pine, by - - 1606 lbs.

Of Norway fir, by	- -	1928 lbs.
Of English oak, by	- -	3860 lbs.

Of the Strength of Torsion.

192. By the experiments of Coulomb, the resistance of wires of the same material, to a force exerted to twist them, appears to increase with the fourth power of their diameters. A similar result follows nearly, from experiments by Rennie, on square bars of cast iron. It would also appear, from a simple course of reasoning, that the resistance must diminish with the distance of the point in the rod to which the twisting force is applied, from the place where it is fixed ; for the rod tends, under the action of the force, to form a screw, the distance between whose threads is the same as the distance between the two points. And in the inclined plane which the screw forms when developed, the twisting force will act as if it tended to raise a weight along the surface of the plane, whose altitude is the constant lineal dimension of the base of the rod.

193. We have no experiments on the absolute resistance to torsion : the following are the relative resistanees of different materials deduced from the experiments of Rennie.

Lead,	1000
Tin,	1438
Copper,	4312
Brass,	4688
Gun Metal,	5000
Swedish Iron,	9500
English Iron,	10125
Cast Iron,	10600
Blister Steel,	16688
Sheer Steel,	17063
Cast Steel,	19562

CHAPTER VIII.

Of the Equilibrium of Artificial Structures.

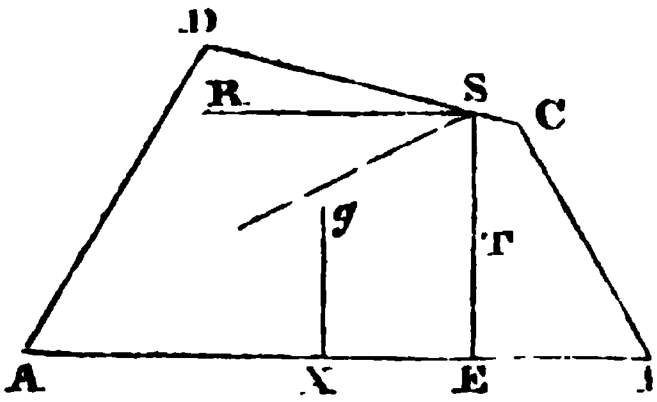
194. Every building, machine, or other artificial structure, may be considered as made up of a system of forces, and in order that it shall be stable, it is necessary, that in this system, the forces which tend to overthrow it shall not exert an effort greater than those which tend to sustain. If equilibrium exist among all the forces that act, the structure will be stable, until some new force be applied to disturb this state of equilibrium; but in order that it shall be permanent, the sustaining forces must have so great a preponderance over those which tend to destroy, that no accidental application of extrinsic force shall be able to overcome the equilibrium. Hence it becomes a matter of great importance in mechanics, that we should be able to state the conditions of equilibrium that exist, among the forces that are to be found in action in those structures which are of most frequent occurrence.

195. When by the action of a disturbing force, any part of a structure is removed from its place, it can only move in one of two ways; it may be pushed directly forwards upon the base on which the structure rests, or upon the adjacent parts of the structure itself; or it may revolve about some fixed point or line. If we call the resultant of the forces, that tend to support a structure, its Strength, and that of the forces that tend to destroy it, the Stress, or Thrust; in the former case, equilibrium must exist between the strength and the stress; and in the latter, between the moments of rotation of these two resultants.

Of the Equilibrium of Walls.

196. A wall may be considered as a prismatic structure; and, in its most simple case, as symmetric on each side of the vertical plane in which the stress acts, and in which its own centre of gravity falls. In this case we may leave the mass of the solid itself out of question, and examine no more than the conditions of equilibrium of its vertical section.

Let ABCD be the vertical section of the wall. Let the stress



S be resolved into two forces, R and T, of which R acts in a horizontal, and T in a vertical direction, it will be obvious that the wall cannot be moved from its place, except by a progressive motion from B towards A, or by a rotary motion around the point A.

The resistance to the former of these, will be the friction. The pressure on the base, AB, will be the sum of the weight of the wall, M, and the force, T, or M+T; the friction then will be

$$f(M+T);$$

and the condition of equilibrium will be

$$R=f(M+T). \qquad (175)$$

To estimate the moments of rotation of the forces: from the centre of gravity, g, let fall a perpendicular, gX, on AD; from S, the point of application of the stress, let fall the perpendicular, SE, on the same line; let $AX=m$, $AE=t$, $ES=r$; the moment of rotation of the weight will be Mm, of the force T, Tt, and of the force R, Rr; the two former will concur to preserve the stability, the latter to destroy it, and the condition of equilibrium will be

$$Rr=Mm+Tt. \qquad (176)$$

If the wall have a rectangular section, and be of homogeneous materials; let its height $=a$, its thickness $=b$, its density $=G$; then, as the mass is the product of the bulk by the density,

$$M=ab\,G. \qquad (177)$$

The distance of the line of direction of the centre of gravity from the point A, on which the wall would tend to turn, will be $\frac{1}{2}b$.

The resistance to a horizontal strain will be

$$abfG; \qquad (178)$$

The moment of the resistance to a rotary motion will be

$$\tfrac{1}{2}ab^2G; \qquad (179)$$

Therefore, in a rectangular wall, the resistance to a horizontal thrust increases with its thickness; and the resistance to an effort to overturn it, with the square of the thickness.

197. In addition to the action of the stress, to move the wall horizontally, or to overturn it, that part of the stress which is exerted in a horizontal direction, tends to break the wall; that part which acts in the vertical direction, tends to crush it. The manner of the action of a force of the latter description has been explained in § 190. Did the wall consist of one piece of a homogeneous substance, the resistance to the former of these forces may be considered as a case of a beam fixed at one end, which has been examined in § 181. But walls are composed for the most part of separate portions of heavy substances, held in their place, partly by the friction of their surfaces, and partly by the tenacity of cements; hence, for the quantity s, in (140), we must substitute the value of these resistances. In respect to both of these circumstances, we have excellent experiments by Boistard, which are to be found in the Treatise of Gauthey "*de la Construction des Ponts.*" By these it appears, that the friction of chisselled stones upon each other is constantly four-fifths of the pressure; that the resistance both of mortar and water cement, is proportioned to the surface, and is equal, in the former, to $1426\frac{3}{4}$ lbs. per square foot, in the latter, to $756\frac{1}{2}$ lbs. When plunged in water, however, the strength of the latter is rather increased than lessened, while the former retains little or no tenacity. The friction, it is obvious, will be greatest in the lowest joints, and in a rectangular wall, will decrease uniformly from the base to the top.

198. When a wall is to resist a horizontal strain, it may be strengthened by making one of the faces sloping, or by building buttresses or counterforts, projecting from it at right angles. The advantages derived from these different methods, may be thus investigated.

(1.) Let p be the base of the sloping part of the wall, being the addition to b of (177) at the base, and on the side opposite to that where the strain acts; the equation (175) will become

$$R=afG(b+\tfrac{1}{2}p)\,; \qquad (180)$$

and the equation, (176),

$$Rr=aG(\tfrac{1}{2}b^2+bp+\tfrac{1}{3}p^2)\,; \qquad (181)$$

If the slope were on the same side with that on which the strain acts, the moment of resistance would be

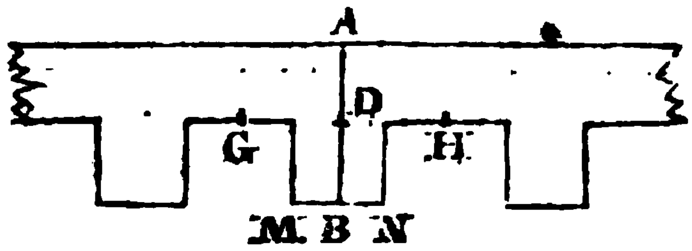
$$Rr=aG(\tfrac{1}{2}b^2+\tfrac{1}{2}bp+\tfrac{1}{6}p^2). \qquad (182)$$

This latter method would therefore be less advantageous than the former. Were the wall to be increased uniformly in thickness, by the quantity $\frac{1}{2}p$, the moment of resistance would be

$$Rr=aG(\tfrac{1}{2}b^2+\tfrac{1}{2}bp+\tfrac{1}{8}p^2)\,;$$

and this would be the least advantageous method of the three.

(2.) Let the figure represent the horizontal plan of the wall



with three of its buttresses, the intervals between which are each divided into two equal parts in the points G, H; if the horizontal strain act uniformly along the whole wall, its resultant will fall in the vertical plane that divides it into two equal parts; and in this section, its centre of gravity will also fall; let this vertical plane be AB; let the height of the wall $=a$, its breadth $=b$; the projection of the buttress, $AD=c$; its thickness $MN=p$; and the interval $GD=d$; the equation for the resistance will become (175)

$$R=afG(bd+cp); \tag{183}$$

for the moment of the resistance, (176),

$$Rr=aG(\tfrac{1}{2}b^2d+bcd+\tfrac{1}{2}c^2p). \tag{184}$$

By an investigation similar to that made in the former case, it will be found, that buttresses applied to the side of the wall on which the strain acts, are less advantageous than those on the opposite side, and that either are preferable to an addition of their quantity of material to a wall of uniform thickness.

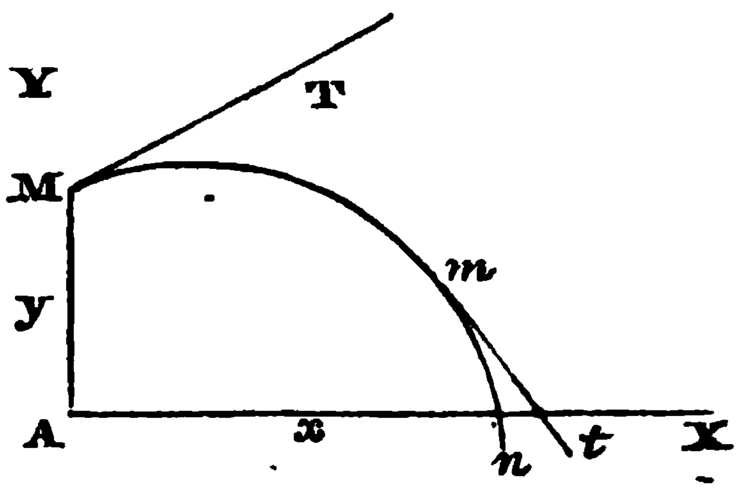
199. These principles are applied to a very great extent in architecture. Thus the walls of the temples of ancient Egypt, which are the most stable buildings containing chambers, that have been erected by the hand of man, and are pressed by heavy stone roofs, have a considerable external slope; the same form is given to the terrace walls of fortifications, which are besides strengthened within by buttresses. In the wonderful buildings of the middle ages, known by the name of Gothic, the walls have such large and numerous apertures as almost to disappear entirely; but in them, the stress of massive vaults of stone is well sustained, by means of buttresses of great projection, and which, by being built with an external slope, unite the advantages of both methods. As an instance of this, may be cited the buttresses of King's College Chapel at Cambridge; these project at their base more than twenty feet from the body of the building, and the whole space between them is almost completely occupied by windows.

Equilibrium of Columns.

200. The reasoning in § 190 and § 191, might be considered as directly applicable to the case of columns having a weight to

support upon their summits. It is not, however, sufficiently strict to be admitted, when the length is great in proportion to the area of the section. A column, as then stated, if its materials be either wholly incompressible, or after they have been compressed as far as their elasticity will admit, can only give way, by a flexure taking place either in the whole mass, or in the fibres of which it is composed. It hence becomes necessary to take into account, the mode in which the stress acts to produce this flexure, and the manner in which resistance of the material opposes it.

Let us take the case of an elastic plate Mmn, infinitely thin,



and fixed at the point M in such a manner, that however it may be bent, the direction of the tangent, MT, shall remain constant. Let us suppose, that this plate is submitted to the action of a number of forces acting in the same plane, which will be that of the curvature. Call the resultant of the components of these forces that are parallel to the axis AY, P, and that of components parallel to AX, Q; the co-ordinates, x and y, of the curve being referred to the same two axes. It is required to determine the conditions of equilibrium of the forces P and Q, and the elasticity of the plate which will tend to restore it to the direction of the tangent MT.

If at any point of the plate m, the part Mn becomes fixed, and the part mn perfectly rigid, a case that will not affect the conditions of equilibrium: the effect of the forces, P and Q, will be to tend to turn the part mn around the point m, and the action of the elasticity of the plate will be exerted to turn the same part of the plate in a contrary direction: hence the elasticity may be considered as a force acting perpendicularly to the line mt, and whose intensity, E, in the case of equilibrium, must be equal to the sum of the moments of rotation of P and Q, in respect to the point m. If p and q, be the distances of these forces from the axes Ax and Ay, their distances from m, will be $p—x$, and $q—x$. The equation of equilibrium will therefore be

$$P(p-x)+Q(q-x)=E. \qquad (185)$$

At any other point than m, the elasticity of the plate will act with a different intensity. The law usually assumed to represent the

elasticity is, that the force is proportioned to the tension, or is at any given point, inversely proportioned to the radius of curvature. If, then, E be the value of the elasticity at a point whose radius of curvature is equal to unity, at any other point the elasticity will be

$$P(p-x)+Q(q-x)=\frac{E}{\rho}. \qquad (186).$$

The well known value of ρ is

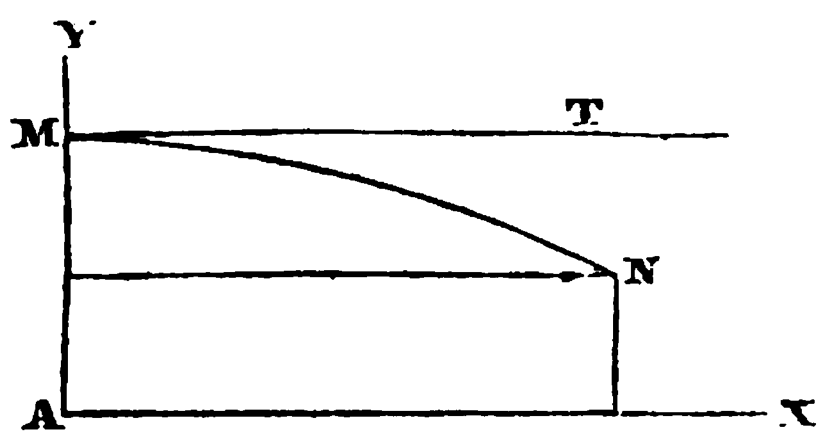
$$\rho=\frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}};$$

whence we obtain

$$P(p-x)+Q(q-x)=\frac{E\frac{d^2y}{dx^2}}{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}. \qquad (187)$$

We may consider this equation with greater ease, by confining ourselves to the cases that may occur in practice.

(1). If we suppose that the plate is so situated that the tangent MT, is parallel to the axis AX, and that the force P is the sole force, and acts at the end N of the plate;



then Q=0, and if we call the length of the plate l,

$$P(l-x)=\frac{E\frac{d^2y}{dx^2}}{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}. \qquad (188)$$

If the curvature of the plate be very small, $\frac{dx}{dy}$ will also become very small, and its square $\frac{dx^2}{dy^2}$ may be neglected in the second member of the equation; we may take therefore

$$P(l-x)=E\frac{d^2y}{dx^2}; \qquad (189)$$

Integrating we obtain

$$P(lx-\frac{x^2}{2})=E\,\frac{dy}{dx}\,; \qquad (190)$$

and there is no need of an arbitrary constant; for when $x=0$,

$$\frac{dy}{dx}=0\,;$$

Integrating again, we obtain

$$y-k=\frac{P}{2.3E}(3lx^2-x^3), \qquad (191)$$

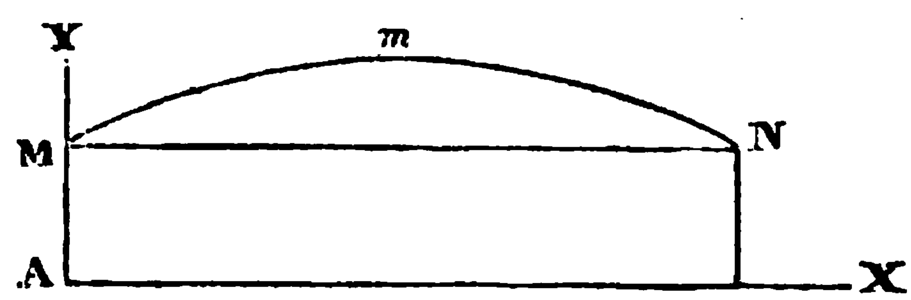
in which k represents the distance AM.

If we make the versed sine of the curvature MB$=f$, when $x=e$, $y=k-f$; and we obtain for the value of f, neglecting its algebraic sign,

$$f=\frac{P}{3E}\,l^3\,; \qquad (192)$$

in which we have a relation between the length of an elastic plate, and the force that is capable of producing a given curvature.

(2). If we suppose that the force P in its turn becomes equal to 0, and that the force Q is applied to the extremity N of the plate, in



such a manner that its prolongation passes through the point M, at which point the plate rests upon a plane AY, the plate will then be curved, as in M m N. If we again neglect $\frac{dx^2}{dy^2}$ the equation of equilibrium will become

$$Q(q-y)=E\,\frac{d^2y}{dx^2}\,. \qquad (193)$$

Multiplying by dy, and integrating we have

$$Q(2qy-y^2+A)=E\,\frac{dy^2}{dx^2}\,. \qquad (193a)$$

In order to determine the constant quantity A, let f be the greatest value of y, and we have at the same time

$$y=q-f\,;$$

and

$$\frac{dx}{dy}=0\,;\ \text{therefore}$$

$$2q(q-f)-(q-f)^2+A=0,$$

and

$$A=f^2-q^2,$$

substituting this value of A in (193*a*) it becomes

$$Q[f^2-(q-y^2)]=E\,\frac{dy^2}{dx^2}, \qquad (194)$$

or

$$dx=\frac{dy}{\sqrt{\left[\frac{Q}{E}\left(f^2-(q-y)^2\right)\right]}};$$

whose integral is

$$q-y=f\sin.\, x\sqrt{\frac{Q}{E}}; \qquad (195)$$

in which there is no need of an arbitrary constant, for when $x=0$, $y=q$.

If we suppose that the axis AX is identical with the line MN, and that the *y*s are positive, when reckoned downwards from this line, then the equation becomes

$$y=f\sin.\, x\sqrt{\frac{Q}{E}}. \qquad (196)$$

In this expression y becomes $=0$, when $x=0$, or $x=l$; and when the curvature is very small, we may consider l as coinciding with MN; upon this hypothesis then,

$$\sin.\, l\sqrt{\frac{Q}{E}}=0;$$

and this can only be true when $l\sqrt{\frac{Q}{E}}$ is an exact multiple of half the circumference of a circle; let m be any whole number, and π, the relation of the circumference of a circle to its diameter; we can express this fact as under

$$l\sqrt{\frac{Q}{E}}=\frac{m\pi}{2};$$

from which we obtain for the value of Q,

$$Q=\frac{m^2\pi^2E}{4l^2}, \qquad (197)$$

and the equation (196) takes the form

$$y=f\sin.\,\frac{m\pi}{2l}\,x. \qquad (198)$$

When $m=1$ which is the smallest possible value,

$$Q=\frac{\pi^2E}{4l^2}, \qquad (199)$$

and

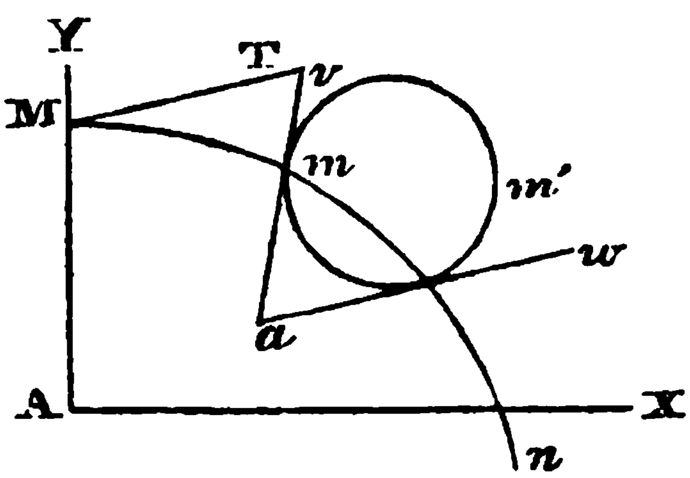
$$y=f\sin.\,\frac{\pi}{2l}\,x; \qquad (200)$$

from these we may deduce that when the value of Q is less than in (199), it will not produce any effect on the elastic plate.

This last case will represent the action of a weight W, upon an elastic plate, standing vertically upon a horizontal plane. We have for the value of the weight that will produce no flexure,

$$W = \frac{\pi^2 E}{4l^2}. \qquad (201)$$

If instead of a single elastic plate of infinite thinness, we consider the forces to act upon a body of definite magnitude; let the forces be as before, P and Q, acting in the plane XY,



parallel to the axes AX AY, let M*mn* be the curvature which the action of the forces has given to one of the generating lines of the surface of the body, and suppose that through some point *m*, of this curve a plane passes, to which the curve at that point is a normal, and that its intersection with the body be represented by a curve such as *mm'*. Refer this curve to two rectangular axes *aw*, and *av*, that lie in its plane, of which *aw* is perpendicular to the plane in which the forces P and Q act, and touches the curve *mm* on the side where the action of these forces tends to bend the body; call the co-ordinates of this curve *w* and *v*.

The body may be considered as made up of an infinite number of elastic fibres; and if we suppose, as before, that the part M*n* becomes fixed, and the part *mn* inflexible, we shall have again for the condition of equilibrium, an equality between the moments of rotation of P and Q, and the force of elasticity. The latter for any one fibre, whose base is $dw \times dv$ is by (186), $\frac{E}{\rho}\, dw\, dv$; its moment in respect to the axis *aw* is $\frac{E}{\rho}\, dw\, vdv$. The moment of rotation of the whole base *mm'* will be $\frac{E}{\rho}\int\int dw\, vdv$, and the condition of equilibrium (186),

$$P(p-x)+Q(q-y)=\frac{E}{\rho}\int\int dw\ vdv. \qquad (202)$$

If the section of the body were a rectangle, one of whose sides

corresponds to av,

$$\iint dw\, vdv = \frac{ab^2}{2}\,. \tag{203}$$

If the section were a circle, whose radius is r,

$$\iint dw\, vdv = \pi r^3\,. \tag{204}$$

If the body were a cylindric tube,

$$\iint dw\, vdv = \pi(R^3 - dr'^2), \tag{205}$$

and the relation between the diameters of a solid cylinder, and a cylindric tube, of equal strengths, would be

$$r = \sqrt[3]{[R(R^2 - r'^2)]}\,. \tag{206}$$

It will be therefore obvious, by comparing the four last equations with (141), (142), (143), (144), and (145), that the resistance to flexure follows the same laws as the resistance to fracture, so far as the strength has reference to the area of the section; but as respects the length, it will be seen by reference to (197), and (199), that the strength to resist flexure, diminishes with the increase of the square of that dimension.

To apply this to the case of a column. Suppose a solid to be placed vertically—that it is charged with a weight W at its upper extremity, and that all its horizontal sections are circular. In this case, W=Q, and if we take for the origin of the co-ordinates the point that bisects the height, $f=q$; the first term in (202) may be neglected. Combining (202) and (204), we obtain for the equation of equilibrium,

$$W(f-y) = \frac{E}{\rho}\,\pi r^3\,. \tag{207}$$

The smallest quantity of material, or the greatest strength with a given quantity will be obtained, when the resistance to flexure is the same in every section. In this case the radius of curvature of the flexure the beam undergoes, is constant, and the curve an arc of a circle, whose radius is ρ.

In a solid of equal resistance, the curve assumed in bending will have a constant radius of curvature, or be a circle whose radius is ρ. The value of y will become

$$y = \rho - \sqrt{(\rho^2 - x^2)}\,;$$

substituting this value of y in the preceding equation, we obtain

$$r^3 = \frac{W\rho}{E\pi}\,[f - \rho + \sqrt{(\rho^2 - x^2)}]\,. \tag{208}$$

This will be a maximum when $x=0$, or the column must be thickest in the middle of its length.

Were this analysis carried farther, it might be inferred that the column should diminish towards each end, to a point, and should therefore have the figure of a spindle.

In the foregoing investigation, however, it has not been taken

into view, that the resistance to a crushing force depends upon the number of fibres, and consequently in a homogeneous body, on the area of the surface pressed. The above inference will, therefore, only hold good in the impossible case, that all the fibres should be curved, and meet at the two extremities in a point.

Let us next take the case of a column resting on a horizontal base, and of a circular section throughout, formed by the revolution of a curve around the axis; and that a force acts, which is distributed in the ratio of the external surfaces of the horizontal sections, and whose directions are all parallel to each other, and perpendicular to the axis.

Call the resultant of these forces R. It is obviously an instance of the case No. (1), of our preliminary analysis.

The equation of equilibrium obtained from (202) and (203) is

$$Rx=\frac{E}{\rho}\pi r^3.$$

If Π be the pressure upon the unit of the surface, we have

$$Rx=\Pi\pi\int r dx=\frac{E}{\rho}\pi r^3;$$

whence we obtain

$$r^2=\frac{\rho\Pi}{3E}x; \tag{209}$$

the generating curve is, therefore, one convex towards its axis, and in which the abscissas are proportioned to the squares of the ordinates.

201. From the preceding course of reasoning it may be inferred:

That the strength of a column is proportioned directly to its area, and inversely to the square of its length; that, were its fibres capable of being collected at each end into a point, it ought to have the form of a spindle. As this, however, is not the case, it may be inferred, that, were reference to be had only to the weight the column is to support, it ought to be somewhat thicker in the middle of its length. As it has also its own weight to support, the swell ought to lie lower than half the height, for in this way the centre of gravity will be lowered, and the base enlarged; and thus the resistance to a force exerted to overturn it, or to one tending to thrust it horizontally from its place, will be also increased.

The practice, then, of architects is founded on reason, and is as follows: In the Grecian Doric, the columns are truncated cones, whose least base is uppermost. In the Roman Doric, in the Ionic, the Corinthian, and Composite orders, the columns are sometimes made cylindrical for one third of their height from the base, and

for the remaining two thirds of their height, are frusta of cones; at other times they swell at one third of their height, the whole external surface being one of revolution, formed by a curve concave towards the axis. It is most usual to make this curve a conchoid.

202. When the weight exerts a lateral thrust, the upper and lower surfaces of the column remaining constant, it may be inferred from §198, that the axis of the column ought not to be vertical, but should be slightly inclined, towards the side against which the weight acts. By a recent examination of the columns of the Parthenon, it has been found that this principle is there applied, and to great advantage. The columns of the external perystyle in this perfect specimen of Grecian architecture, all have their axes slightly inclined inwards.

203. When a column has no weight other than its own to support, but is acted upon by a stress exerted perpendicular to the axis, the form should be that of a conoid, formed by the revolution of a curve convex towards the axis. And which, when the stress is proportioned to the surface, as is the case in the action of a fluid, should be a curve whose abscissas are in arithmetic, and ordinates in geometric progression. We have instances of the application of this principle in the works of nature, in the manner in which the trunks of tree rises from their roots, and their branches from their insertion into the trunk, or into larger branches. In the arts, we find a beautiful application of it, by Smeaton, in the construction of the Edystone Lighthouse. In this remarkable edifice, obstacles of the most appalling character were to be overcome in its erection; and it is frequently exposed to violent swells of the sea. The rock on which it was built is inclined: advantage was taken of this circumstance to cut it into steps, to each of which one of the lower courses of stones is adapted. These steps are so formed, that one at least of the stones of each course is dovetailed to the rock; the remaining stones are so cut, that none of them can be removed, without being lifted, unless the dovetail should be disunited; and throughout all the courses, no one of the outer stones can be removed horizontally without moving all those in the same course. In addition to the tenacity of the cement, the courses were connected by dowels of the form of cubes, of a hard stone, one half of each of which is inserted into each course. This structure has now stood in its perilous situation for seventy years, and has borne without injury, the great stress to which it is exposed. A structure of similar character has more recently been erected on the Bell Rock, at the mouth of the Firth of Forth, in Scotland.

Equilibrium of Terraces.

204. When earth is piled up into mounds or terraces, we may conceive that the faces of these were at first vertical; the earth composing the upper part of the face, being loose, will separate itself and fall; and thus the base will be gradually enlarged, and the face become more and more inclined, until the friction on the surface of the inclined plane, thus formed, becomes equal to the force with which the earth would tend to descend it. At this point, any farther wear of the mound would cease, and it would become stable, each particle on its surface being in equilibrio, under the action of its own weight, the resistance of the surface of which it forms a part, and the retarding force of friction.

In loose soils, the earth will not be supported until the surface make an angle of 60° with the vertical plane; in tenacious soils, the support may take place when the angle reaches 54°.

205. If it be required to support a terrace by a vertical wall, it must be so constructed that it shall be able to bear the horizontal thrust of the prismatic mass of earth, which lies above the plane, that would form the surface of a bank that would be itself supported. But this prism is itself partially supported by friction, and we must, before we can ascertain its horizontal thrust, ascertain how much of the force it would exert in a horizontal direction, is counteracted by the friction.

Let us suppose a weight, W, to lie upon a plane inclined to the horizon at an angle μ; and that a force, A, is applied in a horizontal direction, which, with the friction, just supports the weight. If we resolve each of the forces R and A, into two others; one of which is parallel, the other perpendicular to the plane, they become

$$R \cos. \mu, \qquad R \sin. \mu;$$

and

$$A \sin. \mu, \qquad R \cos. \mu.$$

The two which are parallel to the plane, act in contrary directions; their sum is, therefore,

$$R \cos. \mu - A \sin. \mu.$$

The other two concur in direction; their sum, therefore, is

$$R \sin. \mu + A \cos. \mu,$$

and represents the whole pressure on the plane; the friction will, therefore, be

$$f(R \sin. \mu) + f(A \cos. \mu);$$

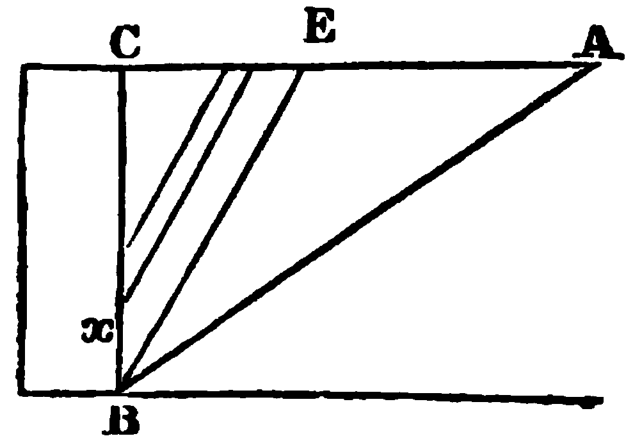
and as the friction is equal to a force that will just support the weight upon the plane, this value of the friction will be equal to that of the forces that act parallel to the plane. Forming an equa-

tion of these two expressions, we obtain from it for the value of A,

$$A=W.\left(\frac{1-f\tan.\mu}{\tan.\mu+f}\right). \tag{210}$$

If, then, the earth be supported by a wall, the second member of this equation will represent the horizontal thrust of the earth.

Let us now apply this to the investigation of the horizontal thrust of the prism, BCE.



Let the angle $CBE=\mu$;
the line $BC=a$;
and the variable ordinate in the direction of $BC=x$.

The element of the surface will be

$$xdx \tan.\mu;$$

and if g be the density of the earth, its weight will be

$$gxdx \tan.\mu;$$

the horizontal thrust of the element will be

$$gxdx \tan.\mu\left(\frac{1-f\tan.\mu}{\tan.\mu+f}\right),$$

or

$$gxdx.\left(\frac{1-f\tan.\mu}{1+f\cot.\mu}\right).$$

If we make

$$\frac{1-f\tan.\mu}{1+f\cot.\mu}=M, \tag{211}$$

the thrust of the element is

$$Mgxdx.$$

Integrating, and making $x=a$, we obtain for the value of the whole thrust

$$\tfrac{1}{2}a^2gM.$$

The moment of the thrust of the element will be

$$Mgxdx(a-x).$$

Integrating, and making $x=a$, we have for the moment of the whole thrust

$$\tfrac{1}{6}a^3gM.$$

The expression, (210), will become $=0$, when $\tan. \mu=0$, or when $\tan. \mu=\frac{1}{f}$; between these two limits there will be a value when the thrust will be a maximum, determined by making $d\text{M}=0$; from this we obtain

$$\tan. \mu=-f+\sqrt{(1-f^2)};$$

if we substitute this value in (210), we obtain for the thrust,

$$\tfrac{1}{2}\, ag \tan.^2 \mu;$$

and for the moment of the thrust,

$$\tfrac{1}{6}\, a^3 g \tan.^2 \mu.$$

The angle whose tangent is $-f+\sqrt{(1-f^2)}$, is just double of that whose tangent is $\frac{1}{f}$, and the latter angle is that at which the earth would just be supported by its own friction; in loose earth, then,

$$\mu=30^\circ, \text{ and } \tan. \mu=\sqrt{\tfrac{1}{3}};$$

in tenacious earth,

$$\mu=27^\circ, \text{ and } \tan. \mu=\sqrt{\tfrac{1}{4}} \text{ nearly.}$$

Hence, in the first, the thrust will be

$$\tfrac{1}{6}\, a^2 g, \tag{212}$$

and its momentum,

$$\tfrac{1}{18}\, a^3 g. \tag{213}$$

In the second, the thrust will be

$$\tfrac{1}{8}\, a^2 g, \tag{214}$$

and its momentum,

$$\tfrac{1}{24}\, a^3 g. \tag{215}$$

It has been already seen, (179), that the moment of the resistance of a wall, whose altitude is a, to a horizontal thrust, is

$$\tfrac{1}{2}\, ab^2 \text{G};$$

and this, in the case of equilibrium, must be exactly equal to the moment of the thrust, or in the case of loose earth,

$$\tfrac{1}{2}\, ab^2\text{G}=\tfrac{1}{18}\, a^3 g; \tag{216}$$

whence we obtain for the thickness of a rectangular wall,

$$b=\tfrac{1}{3}\sqrt{(a^2 g\text{G})}. \tag{217}$$

The value of the thickness of a wall with any given slope, may be in like manner obtained from (181), and that of a wall with buttresses from (184).

Of the Equilibrium of Arches.

206. When an aperture of considerable extent is to be covered by a mass of any material whatsoever, it will appear at once, from what has been said in § 186, that there is a limit to the use of beams or horizontal lintels, growing out of the difference be-

tween the ratios, in which the respective strength, and the action of their own weight to break them, increase. This limit will be reached earlier in stone than in any other material of which we have treated, inasmuch as its respective strength is but small, while its weight is great. So also, in this material, it is frequently difficult to obtain pieces sufficiently large to form lintels, even of a size within the limit at which they would become too weak. In all these cases, we have recourse to what is called an arch.

An arch differs from a lintel, inasmuch as it is composed of a number of pieces of the material, arranged in such a manner as mutually to sustain each other; and as each piece has but small dimensions, measured in a horizontal direction, each will sustain a considerable vertical pressure, while the greater part of the force that is applied to them is borne by their mutual action upon each other's surfaces. Arches, generally speaking, have curved or polygonal surfaces, forming the lower part of their mass, which are concave towards the horizon. They are supported at the extremities upon walls; and in this case, the resistance they oppose to the forces that tend to destroy them, is principally that with which their materials resist a force that tends to crush them. But there are also cases in which the arch is a curve or polygon, convex towards the horizon, in which case their principal resistance is due to the absolute strength of the material.

As both the absolute strength, and the resistance to compression are more intense in all materials, than their respective strength; and as, in addition, the forces that tend to destroy an arch, act in most cases obliquely, it is at once obvious, that an aperture of far greater length can be covered by an arch, than can be done by any other application of the same material.

The circumstances that affect arches will differ according to the materials of which they are constructed. These are principally, stone or brick, cast iron, wood, wrought iron in the shape of chains, and ropes; arches of the three former substances require no distinctive appellation, we shall call arches of the last two kinds, Arches of Suspension.

207. A stone or brick arch is composed of a number of prisms, whose section is a trapezium; these may be considered as truncated wedges, and are called Voussoirs. They are generally of an uneven number; the odd one, which occupies the vertex of the arch, is called the Keystone. The vertical walls on which they rest, are called Abutments, and when there are two contiguous arches, the intermediate wall is called a Pier. The point where the vertical wall meets the curve of the arch, is called the Spring of the arch. The distance between the piers or abutments that support a single arch, its Span. The lower or inner curve

of an arch is called its Intrados; the upper or outer curve, its Extrados. The intervals between the voussoirs are called Joints.

208. In considering the theory of arches, it has been usual to proceed, either by assuming a given intrados, and investigating the relative size of the voussoirs; or assuming the magnitude of the voussoirs, to investigate the curve of the intrados. In both cases, the voussoirs have been considered as wedges, perfectly free to move upon each other, or resisted neither by friction nor the tenacity of cement.

By the former method, it may be shown, that the magnitude of the voussoirs, in order that equilibrium should exist, should be to each other as the portions cut from a horizontal line passing through the vertex of the arch, by the prolongations of the joints. In the case of an arch, whose intrados is a portion of a circle, the relative weights of the voussoirs would be the differences of the tangents; and if the arch were semicircular, the lower voussoirs must be infinite.

By the latter method, it may be shown, that the arch of equilibration, if the voussoirs be equal, must be a homogeneous catenaria; and, in the case of unequal voussoirs, a catenarian curve loaded with weights proportioned to those of the voussoirs. And in the case of weights, varying in the relation of the differences of the series of tangents, the curve would become a portion of a circle.

Both methods being founded upon the same hypothesis, give, not only in the last quoted instance, but in all others, similar results.

An arch of equilibration, determined in either mode, would have its centre of gravity in the highest possible position, and would therefore be in a state of tottering equilibrium. Hence, any action, however small, exerted upon it, would shake the voussoirs from their position, and the arch would be destroyed. So far is this from being the case in practice, that arches may be considered among the most permanent of structures; hence, it is obvious, that so far from the friction and the adhesion of the cements being quantities that may be safely neglected, or for which a mere correction may be applied to an hypothesis from which they are first abstracted, they constitute, in truth, forces as essential to the conditions of equilibrium, as the mutual pressure of the voussoirs themselves.

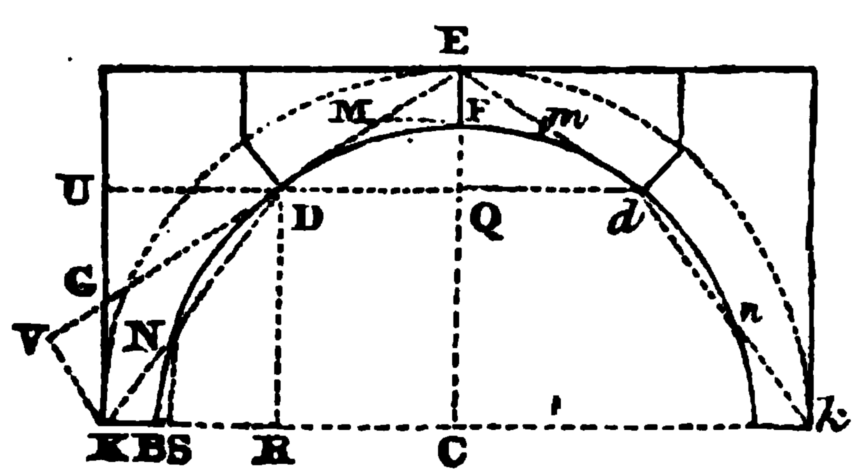
It would also appear from these hypotheses, that an arch would exert no horizontal thrust upon its abutments, unless the face of the abutment were not a tangent to the arch at its spring; and hence, that a thrust would not be created by flattening the arch, provided that the radius of curvature at the spring coincided with

the horizontal line passing through it. All these inferences are, in like manner, contradicted by experiment.

In order, then, to the establishment of a true theory of arches, it is necessary that experiment and observation should be previously called in, to show the exact circumstances under which arches change their form, or actually give way. Such experiments were first made on very small models, by Davisy, at Montpelier, about the year 1732; and were repeated on a larger scale by Boistard, in 1800. Similar experiments were also made by Gauthey; and the same author has recorded his observations upon a number of broken bridges, and others in such a state of decay as to require their being taken down. Perronet likewise, at a previous date, had made accurate observations upon the change of figure undergone by new arches, at the moment of removing their centres.

209. Arches are built by laying their voussoirs upon a temporary arch or frame of wood, called a Centre, whose upper surface has the form it is intended to give to the arch. In laying the voussoirs, it is found that the lower ones retain their position simply by virtue of the friction upon the faces on which they rest, and that they may be laid wholly independently of the centre. They do not begin to slide, until the inclination of the faces becomes equal to about 40°. At this time they begin to press upon the centre, which would have its form changed in consequence, were it not loaded at the summit, for the purpose of counteracting this change of figure.

As the number of voussoirs increases, a pressure begins to take place on the upper part of the centre, which tends to press it against the spring of the arch; the upper voussoirs tend to turn around their lower angle, and the joints open at the extrados. When the keystone is placed, and the centre removed, the open joints close again, and new openings and motions appear in the arch, to represent which, we must have recourse to a figure:



The upper parts of the arch, from D to *d*, are no longer supported, except by their mutual pressure; this tends to close their

joints at the extrados, and press them from each side towards the point E; this point becomes a point of support for both halves of the arch. The joints of the upper part of the intrados will open. The pressure on the point E, necessarily reacts towards the abutments and the lower parts of the arch, which it tends to overthrow by causing them to turn around their outer angles K and *k*. In consequence of this pressure and reaction, each half of the arch separates into parts at some intermediate points, D or *d*. These serve as points of support for the higher parts, and to transmit their action towards the abutments. If the latter do not possess sufficient stability to resist the pressure of the arch, it separates into four parts, which, in breaking, turn around the points K, D, E, *d* and *k*, as upon hinges. If the abutments are capable of meeting the pressure, it still manifests itself by closing the joints of the extrados near the point E, and of the intrados near the points *d* and D; causing those of the intrados to open near the point E, and of the extrados near the points D and *d*.

The position of the points *d* and D, which are called Points of Rupture, depends upon the figure of the vault, and the distribution of the weight it supports. In determining the strength of an arch, it is important to know their position. It is always at the weakest part of the arch; but this is not necessarily that where it is thinnest, as we shall see hereafter.

In the experiments it was found, that:

In semicircular arches, that did not rest on abutments, the points of rupture were at an angle of 30° from the spring.

In oval arches, formed of three circular arcs, at 50°, of the small arc rising from the spring.

In flat arches, the points of rupture were at the spring, as they were in all circular arches whose versed sine was less than a fourth part of the chord.

In all cases the whole mass, say the arch and its abutments, tended to divide into four parts, turning upon points in the intrados, at the spring or base of the abutment, and the vertex, and separating at two intervening points.

210. To investigate the action of the forces that tend to produce these motions.

Suppose the arch to be divided into two symmetric parts, by the vertical plane passing through the line EC, the relations of the forces will be identical on each side of EC. Let K be the origin of the co-ordinates; let x and y be the horizontal and vertical co-ordinates of the point D; x' and y' the co-ordinates of the point E.

Resolve the forces that act at D into two, X and Y, parallel to the two co-ordinates; those that act at E into two also, X′ and Y′.

By the principle of vertical velocities, § 70, the equation of equilibrium will be

$$Xdx+Ydy+X'dx'+Y'dy'=0. \qquad (218)$$

The forces that act at D, act upon a lever KD, which we shall call f, those at E upon another DE, which we shall call g, their values in terms of the co-ordinates are

$$f=\surd(x^2+y^2),$$
$$g=\surd[(x'-x)^2+(y'-y)^2];$$

the differentials of which are respectively

$$xdx+ydy=0,$$

and

$$(x'-x).(dx'-dx)+(y'-y).(dy'-dy)=0.$$

Multiplying the first members of these equations respectively by the constant co-efficients l and l', we have

$$lxdx+lydy;$$

and

$$l'(x'-x)dx'-l(x'-x)dx+\&c.$$

If these be added to the equation (218), we shall find from it that the sum of the terms, that involve the differential of any one of the co-ordinates $=0$, or

$$Xdx+lxdx-l'(x'-x)dx=0;$$
$$Ydy+lydy-l'(y'-y)dy=0;$$
$$X'dx'+l'(x'-x)dx'=0;$$
$$Y'dy'+l'(y'-y)dy'=0;$$

dividing by dx, dx', dy and dy', we obtain

$$X+lx-l'(x'-x)=0,$$
$$Y+ly-l'(y'-y)=0,$$
$$X'+l'(x'-x)=0,$$
$$Y+l'(y'-y)=0;$$

and eliminating l and l',

$$\left.\begin{array}{l} X(y'-y)-Y(x'-x)=0, \\ (X+X')y-(Y+Y')x=0. \end{array}\right\} \qquad (219)$$

The forces which act are the weights of the two portions of the arch, applied to their respective centres of gravity, M and N; call these weights m and n.

The arch being in equilibrio,

$$X=0.$$

If for m we substitute its two parallel components acting at D and E, their magnitudes will be

$$\text{at E,} \quad m\frac{FQ}{EQ};$$

$$\text{and at D,} \quad m\frac{\text{EF}}{\text{EQ}}=\text{Y}'.$$

The force m resolved in like manner, gives for its action, at D,

$$n\frac{\text{KS}}{\text{KR}};$$

hence the sum of the forces that act at D, is

$$\text{Y}=m\frac{\text{EF}}{\text{EQ}}+n\frac{\text{KS}}{\text{KR}};$$

and

$$\text{X}'=m\frac{\text{FQ}}{\text{EQ}}\cdot\frac{\text{DQ}}{\text{EQ}}, \tag{219 a}$$

because it is only the component of $m\frac{\text{FQ}}{\text{EQ}}$ in the direction of E that acts. Substituting these values in the two equations, (219) we obtain identical values for both terms of the first member of the first; in the second, we have

$$m\frac{\text{FQ}}{\text{EQ}}\cdot\frac{\text{DQ}}{\text{EQ}}.\text{KU}-n.\text{KS}-m.\text{KR}=0. \tag{220}$$

In order, then, that equilibrium shall exist, the first term must be equal to the sum of the two last; and for stability it ought to be greater.

The deductions from this theory are abundantly simple.

From the value of the horizontal thrust, X′, it appears that it increases with the length of the line FQ, or with the approach of the centres of gravity of the upper parts of the arch to their summit.

We may, therefore, see that it would be possible to investigate a curve for the extrados, when the intrados is given, that would form an arch without any horizontal thrust whatsoever. When the centre of gravity of the higher part of the half arch falls in the middle, which it will very nearly do, when the arch is very low and the vault of uniform thickness,

$$\text{FQ}=\frac{\text{EQ}}{2};$$

and the formula becomes

$$\frac{m}{2}\cdot\frac{\text{DQ}}{\text{EQ}}\cdot\text{KU}-n.\text{KS}-m.\text{KR}=0. \tag{221}$$

If the arch be flat, EQ becomes equal to the thickness of the voussoirs. If we call this thickness e, and the span of the arch $\frac{l}{2}$, the expression will become

$$\frac{m}{2}\cdot\frac{l}{e}\cdot\text{KU}-n.\text{KS}-m.\text{KR}=0. \tag{222}$$

From this it follows, that the horizontal thrust of a flat arch, is equal to the fourth part of its weight, multiplied by the ratio of the length to the thickness. Not only does the horizontal thrust diminish with the approach of the centre of gravity of the upper part of the half arch to its vertex, but the same causes a diminution in the pressure on the keystone.

In very low arches, this pressure becomes equal to the eighth part of the weight of the whole arch, multiplied by the relation of the span to the whole height of the arch, measured from the top of the keystone.

In flat arches, the pressure on the keystone is just half the horizontal thrust.

211. It will, therefore, be at once seen, that it is advantageous to diminish the depth of the keystone as much as possible; for this will diminish both the pressure upon itself, and the horizontal thrust. No more depth, therefore, should be given to the keystone, than is just sufficient to insure safety from the joint action of the pressure upon it, and the agitation produced by extrinsic causes. The value of the resistance to pressure in various kinds of stone, has been given in §191. It will be proper to allow for its diminution in the ratio of the squares of the depth, which in this case will be the horizontal dimension of the keystone, and to proportion the surface so as to bear three times the weight that is capable of crushing it.

In arches of the form of a circular arc of but few degrees, the horizontal thrust is very great; and even when a sufficient thickness is given to the abutment, absolute safety is not obtained; for the effort of the thrust may be sufficient to overcome the tenacity of the cement, and separate the haunches of the arch from the abutment. This risk is even greater in arches of wood or iron, resting upon stone abutments, and there are several instances on record, of bridges of these materials having fallen as soon as the centering was removed.

The first application of the formula (220), is to the determination of the position of the points of rupture. When the figure of the arch is given, the resolution of this problem presents no great difficulty in its principles. These are as follow: An arch necessarily breaks in its weakest part, which is that in which its resistance has the least ratio to the forces that act upon it, and which is not in consequence necessarily the thinnest part of the vault. The point of rupture will therefore be, where the moment of the force that tends to overturn the lower part, is the greatest possible, when compared with the forces that tend to sustain it.

This ratio is

$$\frac{m\frac{FQ}{EQ}.\frac{DQ}{EQ}.KU}{mKR+nKS};\qquad(223)$$

when this is a maximum, its differential=0.

In circular arches, however, this simple principle is attended with practical difficulties, in consequence of the transcendental quantities which the nature of the circle introduces. In arches formed of several circular arcs, the calculation becomes wholly impossible. In place, then, of a direct mathematical investigation, the position of the points of rupture, determined by the experiments of which we have spoken, is assumed as the basis of the calculation. The resulting thicknesses of the abutments, is considerably less than is usually given in practice, and requires to be increased, in order to allow for want of firmness in the foundation; and to augment the pressure on the plane, where the arch meets the foundation, so much as by the friction to prevent sliding, should the adhesion of the mortar be insufficient. A powerful aid might be obtained in resisting the latter action, by uniting the courses of stone by means of dowels. We give the thickness of abutments, calculated by Gauthey, whose work we have followed in our analysis, for arches of 60 French metres in span; this will serve as a guide in similar inquiries.

Species of Arch.	Thickness of Abutments.	Position of Points of Rupture.
	METRES.	
Semicircular,	1. 32	13° 30′
Oval flattened ⅓d,	1. 62	31° 30′
do do ¼th	2. 24	40° 30′
Circular arc of 60°,	3. 09	0° 00′

212. It has, generally speaking, been usual to give to the piers of bridges the same thickness as to their abutments. But this is by no means necessary; for when two arches rest on the same pier, their horizontal thrusts mutually counterbalance each other; and the pier has no other stress to bear than that of the superincumbent weight. It is frequently advantageous in practice to make the piers as thin as possible; as for instance, when a bridge crosses a rapid stream, or one subject to sudden floods. The ancient bridges, and those of England, generally speaking, have their abutments equal to their piers; and this is absolutely necessary when the arches are built in succession. But when all the arches are carried up simultaneously, and their centres struck at the same time, it becomes practicable to give dimensions to the piers suited merely to the stress they are afterwards to sustain. Such is the present practice of the French engineers.

213. Arches belong as a principal feature to nó architecture more early than that of the Romans. Belzoni indeed states, that he found at Thebes, the relics of arches of brick that seemed to be of a date prior to the Persian conquest. But even were the arch then known, it was but little used. The oldest arch in existence is that of the Cloaca Maxima at Rome, the architects of which were Etrurians; and in the works of the Romans, we find arches superior in magnitude to any that have hitherto been constructed, with but one exception, that we shall presently state. The arches of the bridge of Trajan, over the Danube, were semicircular, raised on lofty piers and abutments, and had a span of 180 feet.

The bridge of Vieillebrioude, in France, approaches more nearly to the bridge of Trajan in dimensions, than any other modern bridge; it was built in 1454, and has a span of 178 feet. And as the bridge of Trajan has long since fallen, it was until within a year, the largest arch in existence.

The bridges of Gignac and Lavaur, in France, have each arches of 160 feet.

The arches of the bridge of Neuilly, and the centre arch of the bridge of Mantes, have each 127 feet span.

The great arch of the bridge of Verona, in Italy, has 160 feet.

The marble bridge at Florence, built by Michael Angelo, has 138 feet.

The span of the centre arch of Waterloo bridge, in London, is 120 feet, and of that of the new London bridge, 140 feet.

At the present moment a bridge is constructing over the Dee, at Chester, in England, whose span is 200 feet. Should this stand after the centre is removed, it will be the greatest stone arch of ancient or modern times.

In consequence of the abundance and excellence of other materials in the United States, our stone bridges are neither numerous nor important. The most beautiful specimen of this species of architecture that we possess, is the acqueduct bridge at the Little Falls of the Mohawk.

Equilibrium of Domes.

214. The same principles are applicable to domes or spherical vaults; and if the curve, in the figure on p. 201, that represents a section of an arch, be now assumed for the section of a dome, the circumstances that take place in its fracture will also be represented, it having points of fracture, and dividing into four parts. But in arches which are cylindrical, or vaults that are cylindroidal, the rupture takes place in the horizontal plane passing through the

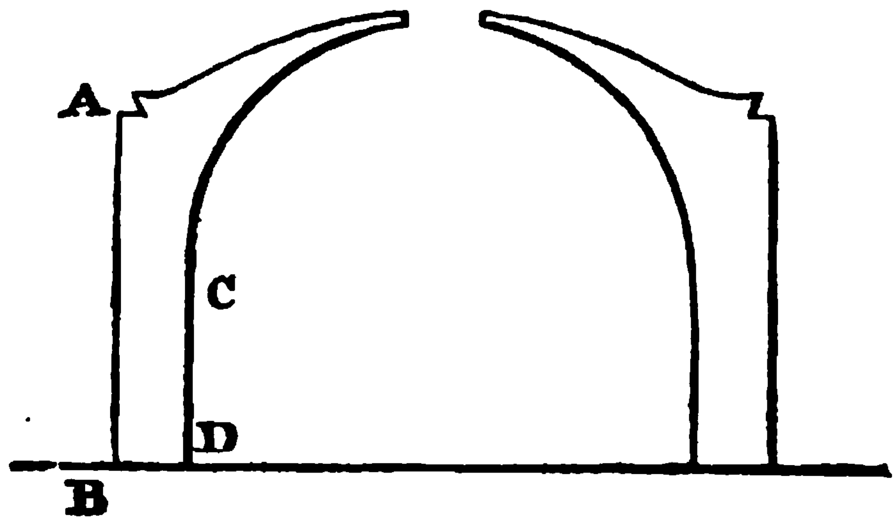
points D and d, while in a dome, the line of rupture is a circle. In domes, the weight of the upper portions will be less than in arches, the points of rupture will lie higher, the horizontal thrust, and the pressure on the keystone will also be less. Domes, therefore, will be borne by less abutments than arches; but as domes are, generally speaking, raised upon a lofty wall, instead of resting upon an abutment, it is inconvenient to give this wall a great thickness; and hence artificial means, that will be presently mentioned, are resorted to, in order to give the requisite resistance.

215. Domes are of more easy construction than arches, for each course keys itself, and they may even be erected without any permanent centre; while an arch must be finished, and the keystone placed, before it supports itself; each course in a dome serves as a keystone to those that are beneath, and apertures of large dimensions may therefore be left in the middle of domes. These serve for the admission of light, or may be made the base of other more elevated parts of the structure.

Domes derive much of their beauty from a geometrical property they possess, which is as follows: the common intersection of a sphere, and a cylinder whose axis is directed to the centre of the sphere, is a circle, and of course a plane curve. Hence, if any number of arches be arranged on the sides of a regular polygon, a dome may be built resting upon them; and a cylindrical tower may be built upon the opening in the centre of a dome, and may in its turn become the support of a second dome.

216. Domes had their origin among the Etruscans, whose temples were circular in plan, and covered with a simple hemispheric vault. The Romans borrowed this species of structure from that neighbouring nation, and brought it to great perfection. The finest antique specimen that remains of this species of building, is that which goes by the name of the Pantheon. This building has a circular ground plan, on which is raised a lofty cylindric wall that bears a hemispheric dome, 144 feet in diameter. As the walls have not of themselves sufficient stability to support with certainty the lateral thrust, they are loaded at the spring of the arch by masonry, accumulated in the following mode: The generating curve of the inner surface, or intrados of the dome, is a semicircle, the

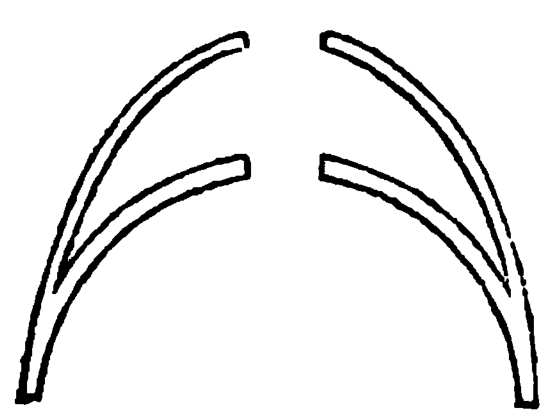
extrados is a less portion of a greater circle, as represented in the



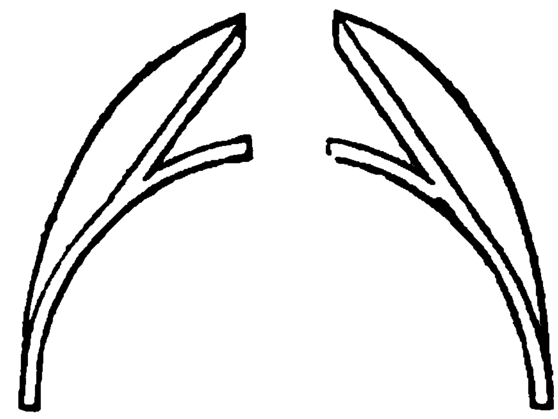
figure; hence it becomes possible to raise the external vertical faces AB, of the wall, much higher than the internal CD; and a great weight rests upon the spring of the arch, while the lower portions of the arch itself are also strengthened.

The domes of a number of Christian churches were built at the intersection of the aisles that form a cross; they were hence borne upon four arches, to which they were applied upon the principle stated in the last section.

To give greater elevation, the first dome was, in the progress of architecture, terminated immediately above the keystones of the arches, and a second dome raised upon the circular ring that constituted the opening. A bolder architect proposed to raise a cylindric wall upon this ring, and support upon it the second dome. Thus was gradually reached the sublime conception of the dome of St. Peter's. In realizing this conception, various practical difficulties presented themselves. The principle applied in the Pantheon was inapplicable, in consequence of the great mass of material it would require, that might have inereased the pressure beyond that which the abutments of the supporting arches could bear. A flattened external dome would have been invisible, except from a distance, and wholly deficient in beauty. For these reasons, the dome that was seen from within, was made of the smallest practicable, but of uniform thickness, and the part seen from without, the half of an oblong spheroid. In the dome of St. Peter's, these two domes are both of masonry, and spring from a common base, as in the figure, diverging as soon as the outer



and innner curve of each, intersect each other. In the case of St. Paul's, in London, the inner dome is of brick, the outer a wooden frame, bearing a covering of sheet lead. To support the frame that bears the outer dome, a truncated cone of brick rises from the inner dome, and bears the smaller cupola or



lantern, which in St. Peter's is borne on the ring that terminates the outer dome. A section of the dome of St. Paul's is represented in the annexed figure.

Although in these different modes, lightness and beauty were both gained, the resistance to the horizontal thrust is so much diminished, that the domes could not have been supported by the balancing of their parts, or by the cohesion of cement; to remedy this defect, the lower courses of the dome of St. Peter's are bound by strong hoops of wrought iron; and at St. Paul's, chains are laid in a groove, cut in the stone ring whence the dome springs, and secured by melted lead.

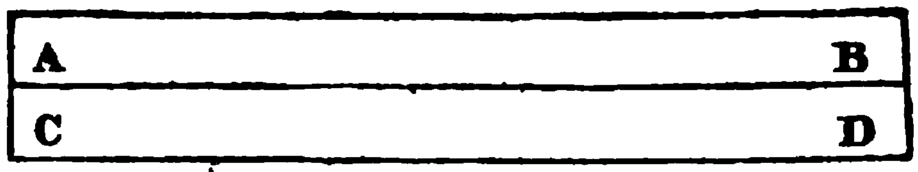
Domes and groined arches are formed in Gothic architecture, at the junction of vaults formed by the intersection of two circular arcs. The lateral thrust is, in these cases, met by buttresses formed on the principles of §198, and the resistance to it is further increased, by loading the buttresses with heavy masses of stone, assuming the form of pinnacles. These, which form one of the chief embellishments of Gothic architecture, are beautiful, not only from their graceful figures, but from their evident adaptation to an important purpose.

Of Wooden Arches.

217. Wood may be applied to the construction of arches also, or to the formation of frames that may answer as a substitute for arches. The application of the principles employed in the preceding chapter, to this material, is, however, different from that which is adapted to the theory of stone arches. While in stone arches the mass is made up of separate parts, which divide in particular points, and move around others as if they had no cohesion, so that the respective strength may be considered as nothing, and the whole available resistances are, that of friction, and that which the material opposes to a crushing force; the use of wood brings into efficient action its respective strength, and the resistance to separation may, in some cases, become that furnished by the absolute strength. The length that can be safely given to a single beam, supported or fixed at each end, and lying in a horizontal position, is limited by its own weight, as has been seen in § 186.

218. If the force that acts be greater than can be borne by a single beam, two may be united in such a manner as to act like

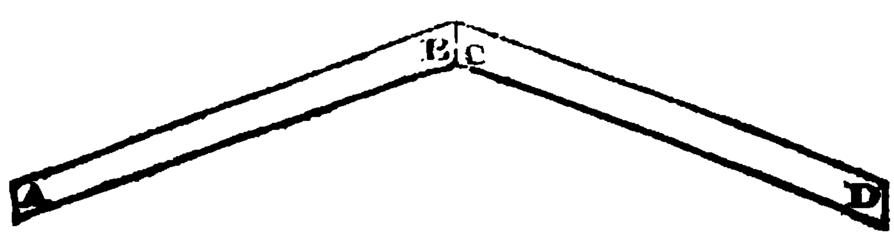
a single piece. Thus, if a beam be merely laid upon another, as



AB upon CD, each will resist the effort to bend it, which must precede fracture, merely with its own force; the system is therefore only twice as strong, in respect to the sum of the forces that act, as the single beam. If then, the single beam have reached the limit at which it breaks by its own weight, they will both be broken. If now the two beams be united, as may be simply done by dowels or pins, in such a way that one cannot bend without causing an equal bending in the other, the two will act as a single beam, and the strength will be four times as great (141), as that of the single beam, or twice as great as the united strength of the two acting separately; hence, the two thus united, will now not only bear their own weight, but require an additional weight equal to their own, to break them. Two beams may be still more advantageously united by cutting their adjacent surfaces into the form of the teeth of a saw, turned in opposite directions, as in the figure. If these be united by screw bolts, or by straps of metal,

both must bend together, and hence act to resist fracture like a single beam. This method is called Trussing.

219. When the limit of strength is reached, either in a single beam or in this arrangement, two beams may next be placed in an inclined position, pressing against each other, as in the figure.



The action of the weight being now oblique to the direction of the beam, the effort of the weight will decrease with the cosine of the angle of inclination, § 184.

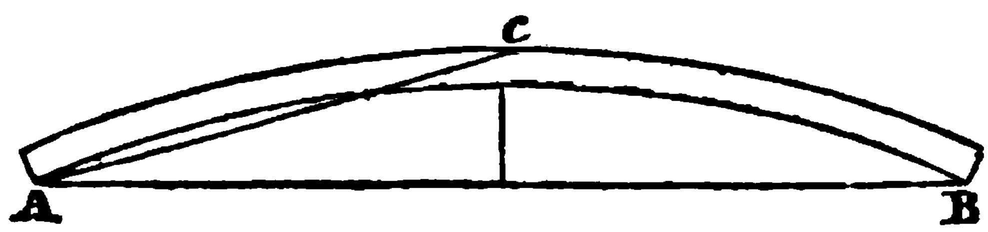
Let F be the strength of the compound beam, and W the weight just sufficient to break it.

$$F = W \cos. i.$$

A lateral thrust will also take place at the points A and B, which may be represented by

$$W \cot. i. \qquad (224)$$

220. If instead of two beams inclined at an angle, a piece of



small curvature be substituted, the effort of the weight will be diminished, § 134, in the ratio of the cosine of the angle CAB; and the resistance of the wood to flexure, as well as its strength, will be enhanced even in a higher ratio, in consequence of the difficulty of bending a curved piece in the direction opposed to its curvature. The latter advantage is of course gained only in the case of the fibres of the wood being also curved; for, if the fibres be cut across, the strength will be diminished; because the lateral cohesion of the wood, which (see § 180) is far less than its respective strength, is now the only resistance that remains.

Whenever the line CA does not cut the intrados, we may without error consider the half of the arch as a straight piece of equal dimensions, loaded at one end by a weight acting vertically, and standing itself upright. This weight will be equal to so much of the force as acts in the direction AC., and the condition of equilibrium, between this component of the whole of the disturbing forces that act, and the strength of the arc will be given by the formula (199).

$$Q=\frac{\pi^2 E}{4l^2};$$

in which l is the length of the arc CA.

We have in the preceding chapter omitted the question of the elasticity of wood, and may, in this case, substitute for that property the respective strength. Using this, it will be obvious that there are three different methods of valuing E, according as the pieces that form an arch, when there are more than one, are combined: when they all act distinctly; when they are merely combined by the vertical posts of the arch, or by other pieces crossing them; and when they are so united as to form a body, no part of which can move without affecting all the rest.

In the first case, supposing each of the pieces to have a rectangular section, whose breadth is a, and depth b; let w be the number from the table of relative strength, n the number of pieces.

$$E=nab^2w; \tag{225}$$

In the last case,

$$E=n^2ab^2w; \tag{226}$$

and in the second case it will be intermediate.

It is, however, hardly possible to unite beams in such a way as

to give them the entire strength determined by the last formula. And in any case whatever, a large allowance ought to be made after calculating the value of E.

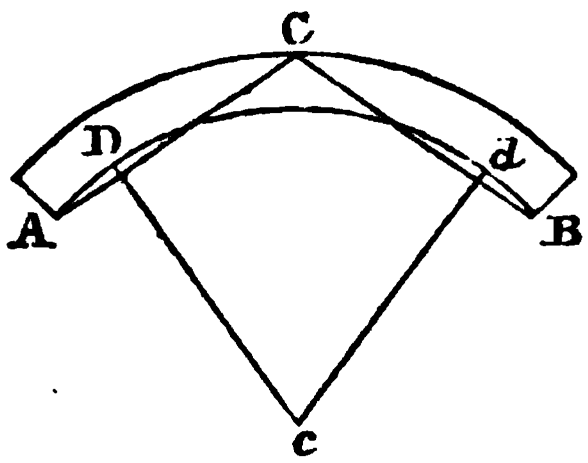
If W be a weight resting upon the vertex of the arch,

$$Q = W \cos. i.$$

But if the weight be uniformly distributed over the arch,

$$Q = \tfrac{1}{2} W \cos. i.$$

221. If, however, the curvature be considerable, we can no longer consider the wooden arc as formed of two straight pieces, but must have recourse to principles in some respects similar to those of stone arches. In the arc ACB, a force acting at C, will not break the beam at that point, but at two points, D and *d*, intermediate between it and the two points of support; and the lower parts, AD and B*d*, will turn around A and B, as in the case of a stone arch, (see § 209). The point of rupture is easily determined in this case; for, the respective strength of a beam of equal thickness, being uniform throughout, if the momentum of the stress be abstracted, whether the beam be straight or crooked, the rupture must occur where the effort of the weight acts most directly upon the beam. This point will be determined by letting fall a perpendicular from the centre of the arc, *c*, upon the line CA.



Having thus determined the point of rupture, we may proceed to determine the conditions of equilibrium between the force P that acts in the direction CA, and the resistance at the point D. This may be done by conceiving that the part DA is immoveable, and that the part DC being firmly fixed at D, is acted upon, by a force that tends to bend it, in the direction CA. This will not affect the condition of equilibrium, and it becomes an application of the formula, (192).

$$f = \frac{P}{3E} l^3$$

In which l is the length of the arc CA, and f the versed sine of the curvature which the force P is capable of producing. From this we obtain for the value of P

$$P = \frac{3Ef}{l^3}. \tag{227}$$

But f is a function of P, and the value of the latter is still involved in the expression, it is, therefore, necessary to have the means of determining f. This can always be safely done by

taking as the value of f, the maximum flexure, or that which precedes rupture. This deduced from experiment, is

$$f=0.0013\frac{l^2}{b}.$$

222. In applying these principles to the construction of bridges, two different methods have been pursued.

(1.) A continuous arc has been formed by bending plank, arranged so that none of their joints should be opposite, and united by bolts and iron straps, in such a manner as to act as a single piece. Such is the principle of the bridges erected in Europe under the direction of Wiebeking. Of these the bridge of Bamberg is the most remarkable. Its span is 221 feet.

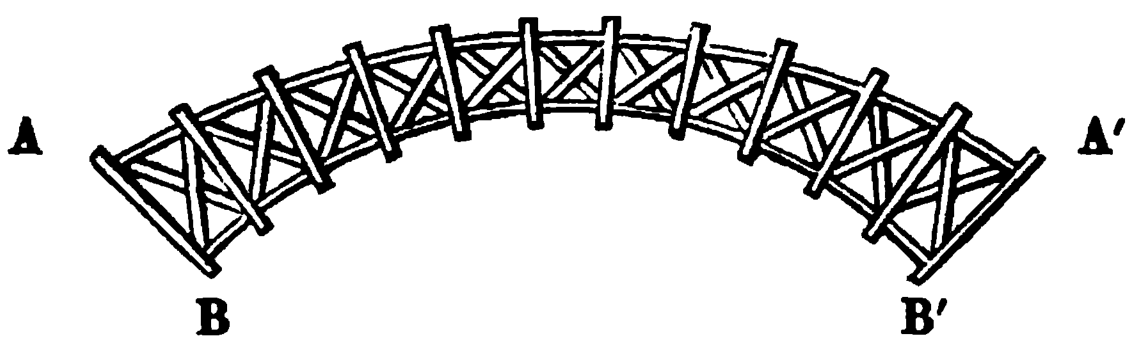
In this bridge, the road passes over the summit of the arch, which is therefore flat, and has a great lateral thrust. If this be not carefully opposed by a proper connexion with the abutments, and by giving them a sufficient weight, the bridge may be destroyed by it. This was the case in two bridges erected on a similar principle at Paterson, New-Jersey.

It is, however, by no means necessary, except where it is desired to leave room for the passage of vessels, to make the road pass over the vertex of the arch. The nature of the material, which is both light and strong, admits of the arch being formed of several separate ribs. Carriages and passengers may pass between these upon a horizontal road, resting on timbers, supported by the arch from above. If the timbers, supported by each rib, be made to form a continuous chord, and be connected with the rib at the spring of the arch, they will, in addition, oppose their absolute strength to the horizontal thrust, which may thus be entirely done away. The ribs, too, may be made with a much greater curvature, and will be both stronger in consequence, and have less horizontal thrust.

Such is the principle of the very beautiful bridge erected by Burr, over the Delaware at Trenton, N. J., the larger arches of which have 194 feet span.

(2.) An arch of timber may be formed of pieces arranged in the form of a polygon; such an arch would be in equilibrio, had it the form of the funicular polygon, § 29; but as the equilibrium would be tottering, it is better to make the system rigid, in which case, it is unnecessary to seek or observe the law of equilibrium. The system may be made rigid by extending some of the timbers beyond the points where they intersect the others, framing them together, and connecting them again with others, forming triangular frames, which cannot alter their shape without breaking. Such is the principle of the wooden bridges of Hampton, and Cambridge, in England. The largest of these, however, has less

than 50 feet span. This method is extremely faulty, and no bridge of any great span constructed after it, has been of long duration. A better mode of making the system rigid, is, to make it double, and interpose, between the arch pieces, queen-posts, AB, A'B', &c. forming by means of mortices and tenants, a series



of open voussoirs, or quadrilateral frames. To prevent these from changing their figure, a frame of the figure of a St. Andrew's cross is placed in each. This principle was adopted in the bridge erected over the Piscataqua, near Portsmouth, New-Hampshire, whose span is 256 feet.

(3.) The two methods may be combined; of this we have an instance in the great arches constructed by Wernwag, one over the Schuylkill, near Philadelphia, the other over the Delaware at Easton. The former has 340 feet span, and is the largest wooden arch now in existence. A project for an arch upon this principle, of 400 feet span, is given by Tredgold, in his work on the principles of carpentry.

223. In all the methods of extending wooden structure across openings, of which we have hitherto spoken, the principle of the arch has been taken as the leading and prominent feature. Far more simple considerations have led to the construction of wooden bridges, of greater span than any we have hitherto cited.

The simplest mode of spanning an opening is a beam; but we have seen that this has an early limit in the size of timber, which cannot be obtained of sufficient depth to enable a long beam to bear its own weight. It might, however, occur, that as it is possible by trussing, as explained in § 218, to obtain beams of greater strength than single trees will afford, so it would be practicable to build a structure which should act upon the principle of a beam. The first thing that ought to be determined, for this purpose, is the figure that would have, under equal size, the greatest degree of strength; such is one that would have the moment of its strength equal in every part of its length.

The beam having a rectangular section, whose constant breadth is a, and variable depth v, the strength of any section, if supported at both ends, will be (152)

$$2av^2;$$

the resistance to flexure will be (203)

$$\frac{E}{\rho} \cdot 2av^2 .$$

If we call the force that acts to bend it, P, we shall have for the condition of equilibrium

$$\frac{Px}{2} = \frac{E}{\rho} \cdot 2av^2 . \qquad (228)$$

If the weight be uniformly distributed, we shall have for the value of P, in terms of the weight w, borne by the unit of the beams' length,

$$P = \frac{wl}{2} ;$$

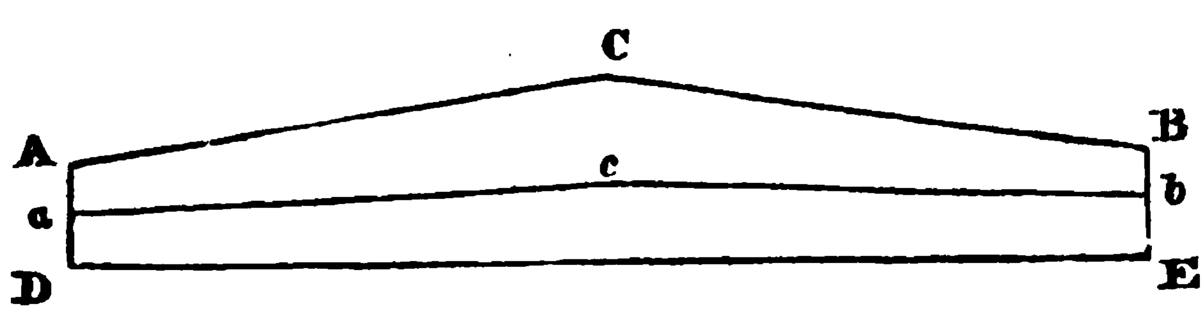
and the equation (228) becomes

$$wx^2 = \frac{E}{\rho} 2av^2 ;$$

whence

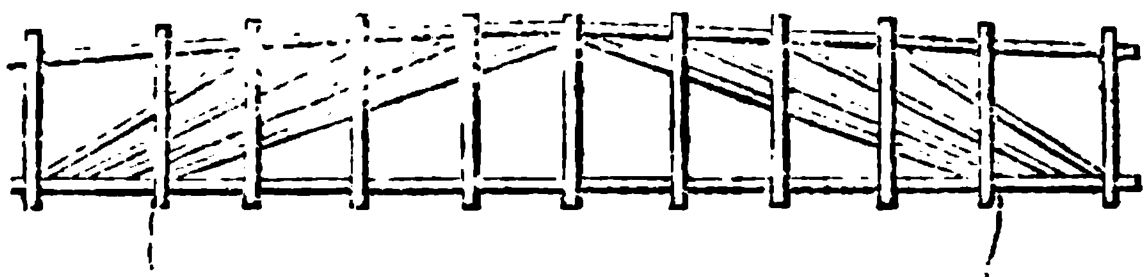
$$v = \frac{x}{2} \sqrt{\frac{\rho w}{Ea}} ; \qquad (229)$$

which is the equation of a straight line, making, with the horizon, an angle, whose tangent is $\sqrt{(\frac{\rho w}{Ea})}$. And as the circumstances are similar on each side of the middle of the beam, the solid of greatest strength is an isosceles triangle. We cannot, however, diminish the thickness of the beam at its points of support to 0, and hence the figure becomes a pentagon, as represented beneath, two of whose sides are parallel and equal, and two of whose angles are right angles.



If we examine the action of the weight to cause this beam to bend, we shall see that the fibres nearest the upper part would be compressed, and those nearest the lower would be lengthened, and thus a line acb might be drawn, which would separate the extended from the contracted fibres. Such a line is called the Neutral Axis. In arranging the pieces of wood of which the bridge is composed, the best method will obviously be, that which shall bring their absolute strength most nearly into direct opposition, to the contractions and expansions, which the beam, if of a single piece, would undergo in bending, but shall which give them

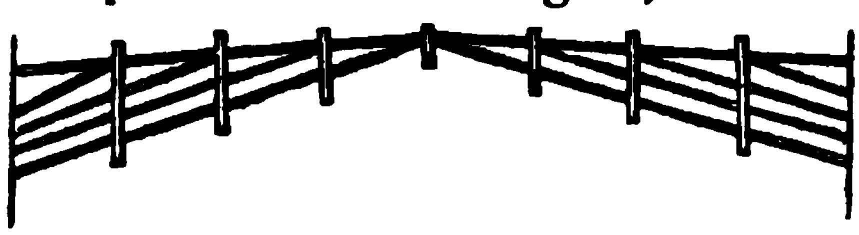
a firm bearing on the abutments. These pieces may then be united by vertical posts, which will form the whole into a series of open triangular frame work; to these posts may be attached a horizontal trussed beam, which will answer to bear the road. Such a plan of frame work is represented beneath, and it will be obvious, that the horizontal beam will destroy the lateral thrust.



This is the principle that was adopted by a Swiss carpenter of the name of Grubenman, in the construction of the bridges of Schaffhausen and Wettingen. The first of these had a span of 365 feet, and appeared to be divided into two spans, resting on an intermediate pier. But the use of this support was in opposition to the desires of the architect, and he had the skill to leave it questionable whether the bridge derived any strength from it or not.

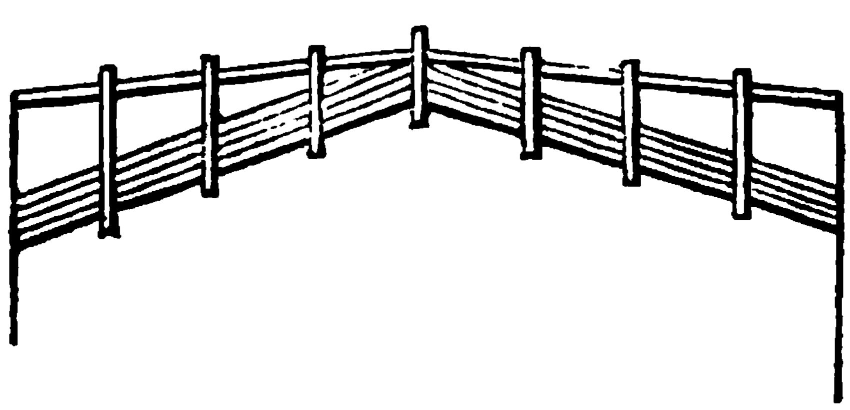
The bridge of Wettingen had a span of 384 feet. Both of these remarkable bridges were of sufficient strength to bear any probable load. Both were unfortunately destroyed during the campaign of 1799, and neither have been replaced.

When from the position of the bridge, the road must pass over its summit, the beam beneath may be suppressed, and the system takes the form represented in the figure, which is that of the



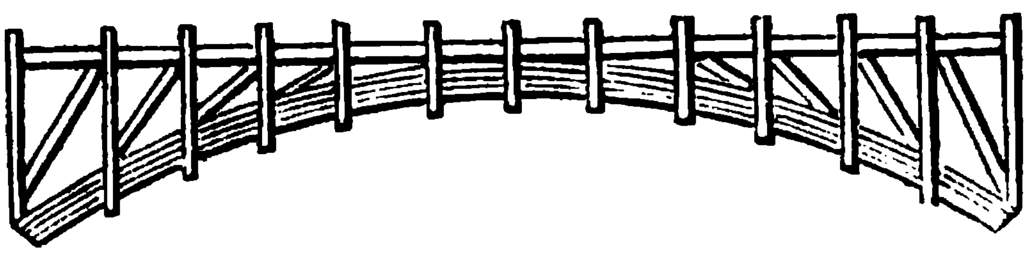
bridge of Kandel, in the canton of Berne, constructed by Ritter. In this there is a horizontal thrust, which must be counteracted by the resistance of the piers.

A modification of the same form, proposed by Gauthey, is represented beneath.

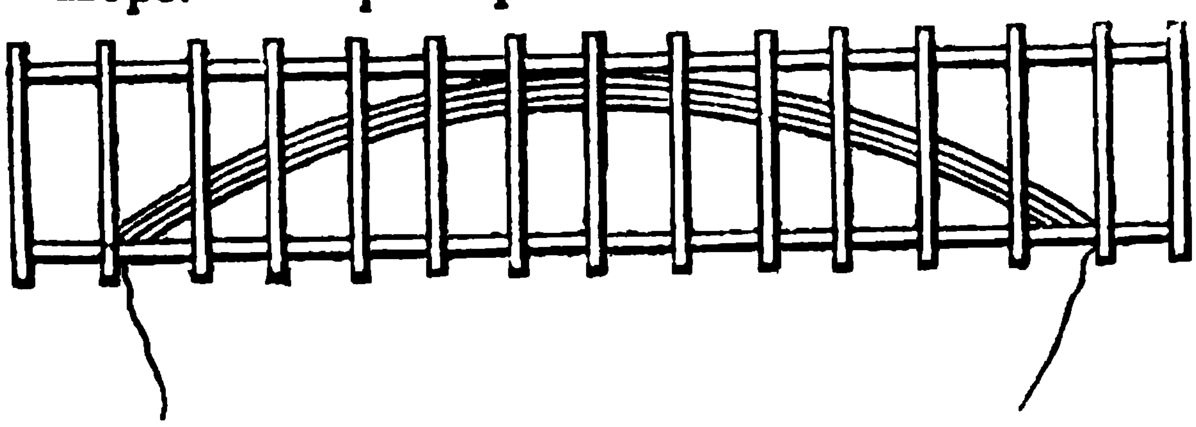


It may be objected to the bridges of Grubenman, that on many

parts of them the moments of the forces that tend to change the figure become very great, in consequence of the great length of the arms on which they act. On this account, a frame of this description has, in many cases, been combined with a bent arc of plank, like that of the bridges of Wiebeking and Burr. When this is introduced, the resistance it opposes to bending seems sufficient, and no other stress need be guarded against except that which tends to absolute fracture. In some of the bridges proposed by Gauthey, therefore, the long pieces that are extended to prevent the former action, are suppressed, and the arch takes the following form:—



A still better arrangement, founded on the same principle, is to be found in the bridge of Wernwag, that crosses the Delaware, at New Hope. The principle of this is exhibited beneath.



An ingenious application of the principle of the beam, has been made by an American engineer, Ethiel Town. His bridge is composed of a rectangular frame of timber, connected by diagonal braces.

This bridge will have great strength to resist fracture, but little to resist flexure, in consequence of the length of the horizontal beams, and the mobility at the angles of the braces. It is, however, lighter than any other plan that has ever been proposed, and might, by a few obvious improvements, be made capable of spanning great openings.

Of Cast Iron Arches.

224. Cast iron bridges may be erected either of continuous ribs and bands, or by forming the material into skeleton voussoirs. In the first case, their theory is the same as that of wooden arches, substituting the resistance of cast iron for that of wood; in the second case, their theory is similar to that of stone arches,

substituting the value of the strength of the one material for the other. Stone, a material of small respective strength, and more than double the weight of wood, requires that the intrados should have a continuous surface, and the space between it and the extrados is often of necessity filled up. Wood, a material of greater respective strength, of less resistance to a crushing force, and less weight, may be arranged in separate and similar ribs, and a lightness in the arch, far more than proportioned to the different densities of the substances, being thus attained, spans of far greater extent may be compassed than by stone. Cast iron holds an intermediate rank between these two materials: being possessed of more strength than either, it has three times the density of stone; but it may, like wood, even if formed into voussoirs, be arranged in separate ribs; the voussoirs, too, need no more material than will form a flaunch around their periphery, and furnish supporting braces within. Still, iron will form an arch much heavier than wood; and the limit at which it is crushed by its own weight is, therefore, sooner reached.

Of the two methods which have been mentioned, that of continuous ribs is objectionable, because cast iron, from its crystalline structure, has no great tenacity; hence voussoirs are better, by means of which the resistance to crushing is substituted for the respective strength.

225. After the full manner in which the theories of stone and wooden arches have been discussed, it is unnecessary to enter into a repetition of the principles, even for the purpose of applying them to another material. We shall, therefore, be content with giving a list of the principal arches of cast iron that have hitherto been constructed.

The first instance of an iron bridge is that of Coalbrookdale in England. Its span is 100 feet. The main rib is composed of no more than two pieces, meeting at the vertex of the arch. It is, therefore, an instance of the first kind of cast iron arch. It was erected about the year 1776. The second cast iron bridge is at Buildwas, near Coalbrookdale. Its span is 131 feet. The date of its erection is 1795.

One at Sunderland, in England, over the Wear, erected in 1796, has a span of 209 feet. It is the first instance of the formation of cast iron into voussoirs.

The bridge of Austerlitz, at Paris, has five arches of 106 feet each, and is composed of skeleton voussoirs, very scientifically arranged.

Of Chain Bridges.

226. Bridges may be supported by means of chains, or ropes.

In the earlier forms, planks were laid directly upon the the flexible material, which was stretched between two firm supports; the chain or rope was, for the convenience of passage, brought into a position as nearly horizontal as possible. This method is, however, inconvenient in practice, and diminishes the resistance of the material; for it will be seen by reference to § 27, that, considering the bridge as a funicular polygon of an infinite number of sides, the effort of a weight acting in a vertical direction to break it, will be much increased by diminishing the curvature; and, after all, the road could never cease to have an inconvenient slope.

A far better application of the principle was carried into effect about the year 1796, by Mr. James Findley of Pennsylvania. Instead of attempting to render the road nearly level by the tension of the chains, he made them of such length that the versed sine should be not less than one seventh of the span. The road was suspended from the chain, and might therefore be rendered perfectly horizontal. Chains, iron rods, or beams of wood may be used to effect the suspension. Forty bridges of this description were erected in the United States previous to the year 1808. As wrought iron is the material that has the greatest absolute strength; as the chains by which the road is suspended may be multiplied, and the longitudinal beams having thus many points of support, need not be very thick; and as wood is the lightest material of which a road can be constructed, it is susceptible of demonstration, that an arch of greater span may be constructed upon the principle of Mr. Findley, than in any of the modes that we have yet spoken of, or indeed in any other manner that has yet been proposed. It is only wonderful that chain bridges have not yet come into more general use, for there are innumerable cases in which they possess greater advantages than those of any other material.

It was not until 1814, that Findley's method attracted the attention of European engineers. At that date it was in contemplation to shorten the post road from London to Liverpool, by effecting a passage across the Mersey at Runcorn, a position in which no other material would have been applicable; for the locality required an arch with a span of 1000 feet. Telford, therefore, proposed a bridge composed of a timber road suspended by chains, identical in principle and form with those of Findley; and having instituted a series of interesting and useful experiments on the strength of wrought iron, he proved beyond all question the practicability of the project. It has not however been carried into effect.

In 1818, a bridge formed of iron wire, bearing a path for foot passengers, which had been erected the year before by the Earl of Buchan, across the Tweed at Dryburgh, was carried away,

and replaced immediately by a chain bridge, which was the first erected in Europe on Findley's principle. About the same period, Telford presented a project for one of similar character, across the straits of the Menai, which has since been executed.

The latter is, in point of extent of span, the most remarkable bridge in existence. The distance between the centre of the piers is 560 feet, and the road is elevated 100 feet above the level of the highest tides. The height of the supports above the level of the road, which height corresponds with the versed sine of the arc, was intended at first to have been no more than 35 feet, but has been increased to 50 feet.

Before the completion of the bridge over the Menai, Captain Brown, so well known as the introducer of chain cables for ships, completed the construction of a chain bridge over the Tweed, near Berwick: this was opened to the public in 1820, and was the first executed in Europe, which was adapted for the conveyance of loaded carriages. Since that period, numerous other bridges, and several piers for the reception of vessels have been constructed in Great Britain, and the method has been successfully introduced into France.

227. A bridge formed of chains with a road suspended from it, is in the condition of the funicular polygon, § 29. Its theory may, however, be more easily reduced to calculation for practical purposes, by considering it as a catenaria, and the inequality in the distribution of the weight is too small to cause any error in practice, from assuming it to be loaded at every point with an equal weight.

If we take one of the points of suspension of the chain, for the origin of the co-ordinates, x and y; and if a be the angle the curve makes with its tangent at that point; w the weight borne by each unit of the length of the chain, assumed to be constant; and T the tension of the chain at the point of suspension, we have by the theory of the catenaria, § 28, for the equation of the curve

$$x=\frac{\text{A}\cos. a}{w}\cdot\log.\left\{\frac{\text{A}-wy\pm\sqrt{[(\text{A}-wy)^2-\text{A}^2\cos.^2 a]}}{\text{A}(1-\sin. a)}\right\}; \quad (44)$$

for the versed sine f, equal to the height of the point of suspension above the level of the horizontal tangent of the curve,

$$f=\frac{\text{A}(1-\cos. a)}{w}; \quad (44a)$$

for the half span, $=\frac{1}{2}s$,

$$\tfrac{1}{2}s=\frac{\text{A}\cos. a}{w}\log.\left(\frac{\cos. a}{1-\sin. a}\right), \quad (44b)$$

for half the length of chain, $=\frac{1}{2}l$,

$$\tfrac{1}{2}\, l = \frac{\text{A sin. } a}{w}\,; \tag{44c}$$

and for the relation between f and l,

$$f = \tfrac{1}{2}\, l \frac{1-\cos.\ a}{\sin.\ a}\,; \tag{44d}$$

from (44*a*) and (44*c*), we obtain for the values of A,

$$\text{A} = \frac{wl}{2 \sin.\ a} = \frac{wf}{1-\cos.\ a}\,. \tag{230}$$

The direct determination of the value of the angle a, from the properties of the catenaria, is not easy; but when the versed sine does not exceed $\frac{1}{10}$th of the span, the curve does not differ sensibly from a circle, that would have the same lines for its tangents at the points of suspension: from the properties of the circle, the value of the tangent of the angle a, is found in terms of its radius R, and chord d, to be as follows, viz:

$$\tan.\ a = \frac{d}{\sqrt{[(2\text{R}+d).(2\text{R}-d)]}}\,;$$

and the value of the radius may, from the properties of the same curve, be found by the subsidiary formulæ,

$$\text{R} = \frac{d^2+4f^2}{8f}\,.$$

These formulæ and principles, will, generally speaking, suffice for the calculation of any of the cases that can occur in practice. But for a more full discussion of the theory in which all the circumstances are taken into account, we refer to the work of Navier, *Rapport et Memoire sur les Ponts suspendus, Paris,* 1823.

228. From this we quote the following practical rules that are immediate inferences from his investigations. In increasing the span of chain bridges, there is no reason to fear that the changes of figure, growing out of the action of passing loads, will be augmented. These changes may even be rendered less sensible, in bridges of wide span, by making the versed sine of the curve bear a less proportion to the extent of span. In fact, as the ratio of the versed sine to the span diminishes, the curve of the bridge will approach more and more near to a straight line, and at this limit the figure of the chains is invariable, whatever be the manner in which the weight is distributed, provided they be, as the hypothesis assumes, inextensible. On the other hand, by diminishing the versed sine, the constant tension, that the weight of the road exercises on the chains, is increased also, as well as the variable tension growing out of accidental loads; and both of these tensions would become infinitely great, if the chain were stretched in a straight line.

229. In calculating the dimensions of the iron chains which compose the inverted arch, as well as those by which the road is suspended from them, the absolute strength, § 179, should be made at least twice as great as the greatest probable tension; for an iron rod will, at a limit of strain, a little greater than the half of its absolute strength begin to stretch; and its elasticity will be so much impaired by the strain, that it will not restore itself to its original dimensions.

230. Iron is subject to fracture on sudden changes of temperature. This appears to arise from unequal expansion; the outer parts being sooner affected, expand or contract earlier than those within. To prevent any danger from this cause, heavy loads should not be permitted to pass, for some hours after any great and sudden change in the temperature of the air shall have occurred. As changes sufficient to cause danger are by no means frequent, such precaution cannot be productive of any important inconvenience.

231. Bridges formed of chains are liable to two species of oscillations: the one vertical, growing out of the passage of weights; the other horizontal, arising from the action of the wind, or other external disturbing forces. In respect to the former, they are more likely to produce injury in small than in large arches, for both the extent and velocity of the vibrations decrease with the increase of the span; the extent of this kind of oscillation follows the inverse ratio of the squares of the chord of the curve; while the velocity decreases with the length itself.

232. Had the chains no weight to support besides their own, they would be readily caused to oscillate in a horizontal direction, and would follow in their vibrations the law of pendulums, the time of performing them being independent of the intensity of the disturbing force. The mathematical investigation shows, that the amplitude of these oscillations would diminish in a ratio more rapid than that in which the length of the chains increases. The chains, however, are connected with the timber road in such a manner, that they cannot oscillate in a horizontal direction without causing it to change its figure, either horizontally or vertically. As it is easy, by a proper system of framing, to render the road almost inflexible in a horizontal direction, these oscillations can be at most but small, unless the disturbing force become of sufficient intensity to cause the rupture of the wood work. So long, then, as the tendency to oscillate is but small, it may be performed without meeting with much resistance; but so soon as it begins to increase, it is opposed by the whole rigidity and weight of the timber road.

233. The strength of wood, when drawn in the direction of its fibres, being very great, amounting, § 180, to about one-eighth part of the strength of wrought iron, the former material may be used to suspend the road, from the principal chains, to great advantage, particularly as regards economy of construction.

234. The expense of the chains is a minimum, when the versed sine of the catenaria has to its chord the relation of $1 : 2\sqrt{2}$; but this is a proportion that is rarely admissible in practice.

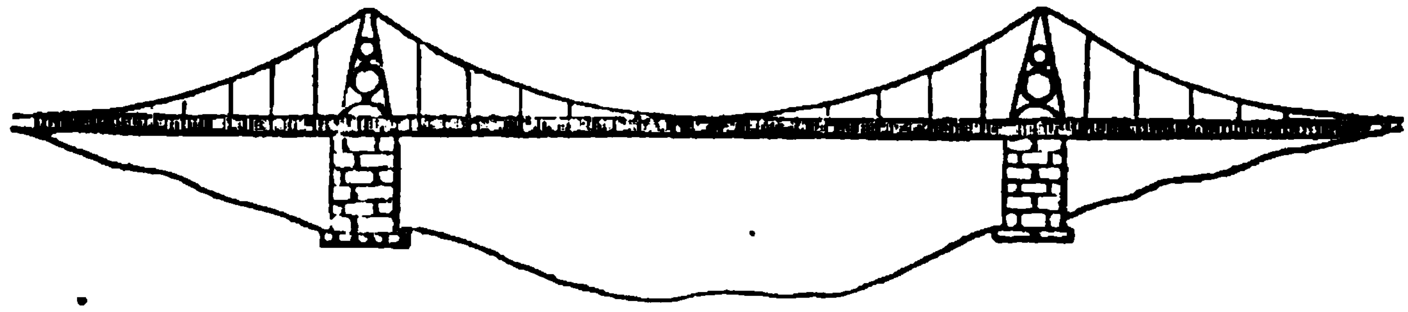
235. The establishment of chain bridges offers a great variety of different cases, that may comport with arrangements more or less simple in their structure, thus:

(1.) The bridge may cross a ravine situated in a passage enclosed between rocks, that are high enough to afford firm and fixed points for fastening the chains; in such a case, and particularly when the breadth of the gorge exceeds 4 or 500 feet, a chain bridge may not only be the most economical, but often the only practicable method of passage.

(2.) The space to be traversed by the arch, may offer firm supports for the chains, at a sufficient height on one side only. In this case the curve may advantageously have the form of a half catenaria, being attached at a proper height to the lofty bank, and having for its tangent at the opposite bank, a horizontal line.

(3.) When both banks are low, the chains must be attached to artificial supports. These are sometimes masses of masonry; at others, frames of cast iron; and if they are not themselves sufficiently solid to sustain the tension, they must be reinforced by chains extending from them, in directions opposite to that in which those which support the bridge are suspended, and which must be firmly fastened to the ground.

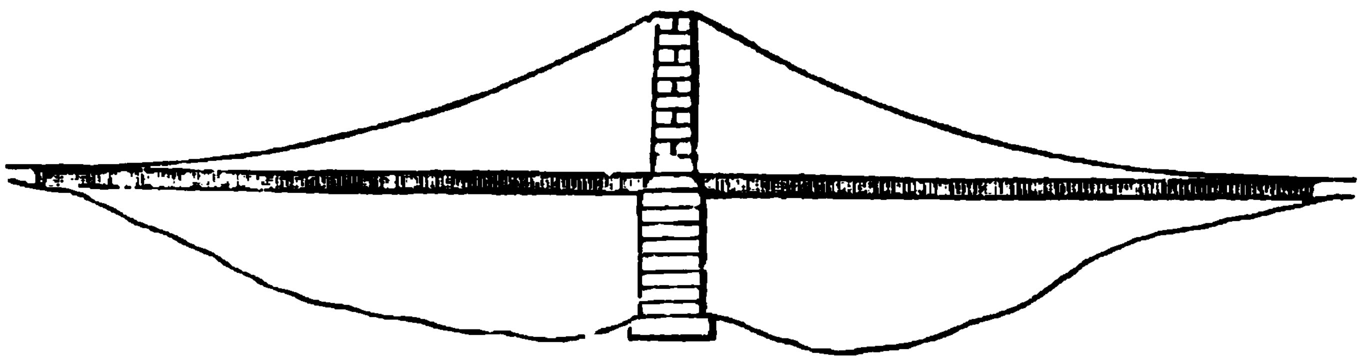
(4.) This last mode may, in some cases, be modified to advantage, by advancing the piers, or artificial supports, into the space that is to be traversed. In this case, the chains that strengthen the piers are made to bear a portion of the bridge, and the whole may assume the arrangement figured beneath, of one whole and



two half arches. This was the plan proposed by Telford, for the bridge at Runcorn.

(5.) A single pier may, in some cases, be erected in the middle of the space, and the bridge take the form of two half cate-

nariæ, as represented in the figure. A bridge of this shape was



constructed in England, under the direction of Brunel, to be erected in the Island of Bourbon.

When the extent of the proposed bridge is great, various combinations of whole and half arches may be formed, according to the local circumstances.

The theory would show that, while the height of the points that support the chain may be increased indefinitely, there is no practical limit to the extent of the span, except that at which the chains would, if suspended vertically, break by their own weight; for additional strength may be gained by increasing the proportionate magnitude of the versed sine of the curve. This investigation is however of little value in practice, for besides the immense expense of artificial supports of great height, the oscillations are rendered, as has been seen, more intense by such an increase. For this last reason, the proportion originally employed by Findley of $\frac{1}{7}$, has been reduced to $\frac{1}{10}$ in most instances.

If the proportion between the span and the versed sine is constant, the length of the former has a theoretic limit. This has been calculated by Navier, under the assumption that the relation is $\frac{1}{15}$, and found to be 2900 feet. As no bridge has yet been erected of a span as great as 600 feet, we are probably still beneath the practical limit. It cannot, however, be advised to attempt the construction of chain bridges much exceeding the last mentioned extent, and the increase of the span will probably be made by gradual steps, as has been the case in other instances.

236. Bridges of wire, and round iron of different sizes, have also been used; in these the road rests directly upon the wires. More recently, in France, Seguin has proposed to use iron in this form, and to suspend the road from it, as in the chain bridges of Findley. He has urged in favour of his proposition several plausible arguments, among which are, the greater strength obtained by an equal quantity of metal in this form, and the entire removal of the risk that arises in chains from an imperfect welding of the links.

BOOK IV.

OF THE MOTION OF SOLID BODIES.

CHAPTER I.

Of Falling Bodies.

237. From what has been said in the preceding book, it will be seen, that, when a force whose direction passes through the centre of gravity, acts for an instant of time upon a body, and then abandons it to itself, the motion is exactly such as it would have were all the matter of the body collected in that centre, and the force were to act there with an intensity equal to that which it actually has.

We, in truth, know of no such forces in nature. No body can instantaneously acquire the velocity due to the force impressed, but must pass through inferior degrees of motion, requiring a certain time for the purpose. But there are innumerable cases, where this time is so small as to be absolutely inappreciable, and we may therefore, without committing any error, assume that the action is instantaneous.

So, also, as every body is composed of a number of separate particles, which in solids are united by the attraction of aggregation, (a force that however intense, does not render the system absolutely rigid or invariable,) a greater or less time will be required to convey the impulse from the point to which it is originally applied, and to distribute it throughout the system. This time is, like the other, so small as to be inappreciable; still there are many cases where we see sure indications of the motion having been communicated by degrees; and there are even some, where we take advantage of this circumstance in our practical applications.

The centre of gravity of any body acted upon by a force, the duration of whose action is so small as to be insensible, and whose direction passes through that point, moves uniformly forwards in the straight line which marks the direction of the force. If two such forces act, the centre of gravity moves uniformly in

the diagonal of a parallelogram, whose sides are parallel to the direction, and represent in magnitude the intensity of the forces. And thus of any number of forces, according to the principles of § 42.

238. When a body falls near the surface of the earth, it is acted upon by accelerating forces, whose directions are perpendicular to that surface. Within the small space that any body, whose motion can usually become the object of investigation, occupies, these forces may be considered as parallel to each other. Their resultant, then, will pass through the centre of gravity, to which we may therefore conceive, that a single accelerating force is applied.

The attraction of gravitation to which this action is due, varies (§ 100) at different points of the earth's surface, and decreases as we rise from the surface, (§ 96), according to a determinate law. Both of these circumstances may also be neglected without causing any sensible error; and hence, a body left without support, near the earth's surface, may be considered as a body moving from rest, under the action of a constant accelerating force. It will therefore move in a straight line, in the direction of the force, or perpendicular to the surface of the earth, and with an uniformly accelerated velocity. All the formulæ of § 49, are therefore applicable to this case, provided a value be assigned to g, the measure of the accelerating force.

By the methods described in Book III., Chap. I., and others to which we shall hereafter refer, it has been inferred, that a body moves from rest in a second of time, under the action of the force of gravity, through a distance of $16\frac{1}{12}$ feet. Hence we have for the value of g, (60), $32\frac{1}{6}$ feet.

In most of the calculations in which the formulæ of § 49 are employed, it is sufficient to take the approximation of

$$g = 32 \text{ feet.}$$

Substituting this value we obtain

$$\left.\begin{aligned} s &= 16t^2 = \frac{v^2}{64} = \tfrac{1}{2}tv, \\ t &= \frac{v}{32} = \tfrac{1}{4}\sqrt{s} = \frac{2s}{v}, \\ v &= 8\sqrt{s} = \frac{2s}{t} = 32t. \end{aligned}\right\} \qquad (231)$$

239. A heavy body projected upwards from the surface of the earth, will be retarded by a constant force, which will finally destroy the upward motion. It will then begin to fall; the motion upwards will be equably retarded, the motion downwards equably accelerated.

For the time of flight, and the height to which it rises, we have, by the substitution of the same value of g, from (61 a) and (61 b),

$$t = -\frac{v}{32}, \qquad h = -\frac{v^2}{64}. \tag{232}$$

Hence the body will rise to the height whence it must have fallen, in order to acquire the initial velocity; and the times of ascent and descent will be exactly equal. The whole time of flight will be $\frac{v}{16}$.

240. When the motion of bodies falling near the surface of the earth is actually observed, it is found to differ materially from what would be obtained from the calculation of the preceding formulæ. Thus Dr. Desaguliers found that a ball of lead of two inches in diameter, took $4\frac{1}{2}$ seconds to fall from the dome of St. Paul's, in London, to the pavement, a distance of 272 feet. Now by the formula (231),

$$s = 16t^2.$$

It ought therefore, in the same time, to have fallen through 324 feet. In a body of less diameter and similar density, and in one of equal diameter and less density, the difference between the formula and the experiment would have been still greater. The cause of this difference is the resistance of the air, a retarding force analogous to friction, but which follows a different law.

In a subsequent part of the work, we shall examine the nature and character of this resistance. For the present, it is sufficient to state, that it is usually assumed to increase in the ratio of the square of the velocity. This is nearly true when the distance through which a body falls does not exceed a few hundred feet: and it therefore may not be wholly useless to investigate the motion of falling bodies upon this hypothesis.

Taking the notation of § 237, the resistance of the air may be expressed upon the above hypothesis, by multiplying v^2 by a constant co-efficient.* Call this co-efficient gk^2, the retarding force will be,

$$gk^2v^2;$$

the accelerating force of gravity being g, the actual force, f, which accelerates the body's motion, will be

$$f = g - gk^2v^2;$$

or

$$f = g(1 - k^2v^2).$$

From (53) and (54), we may deduce the equations

$$fdt = dv, \text{ and } fds = vdv,$$

* Venturoli, vol. I. p. 89

which become

$$\left.\begin{aligned} gdt &= \frac{dv}{1-k^2v^2}; \\ gds &= \frac{vdv}{1-k^2v^2}. \end{aligned}\right\} \tag{233}$$

Integrating, and remarking for the determination of the constant quantity, that when $t=0$, we have at the same time $s=0$ and $v=0$, we obtain the two equations,

$$v=\frac{1}{k}\cdot\frac{e^{2gkt}-1}{e^{2gkt}+1}, \tag{234}$$

$$s=\frac{1}{2gk^2}\cdot\log.\frac{1}{1-k^2v^2}; \tag{235}$$

and eliminating v, obtain a third,

$$s=\frac{1}{gk^2}\cdot\log.\tfrac{1}{2}\left(e^{gkt}+e^{-gkt}\right). \tag{236}$$

From equation (234) it will be seen, that the greatest possible value of v cannot exceed $\frac{1}{k}$. Hence, if the body fall from a considerable height, the velocity may finally become uniform.

The motion of a rising body might be investigated, by taking for the retarding force f,

$$f=g+gk^2.$$

The constant co-efficient, gk^2, is found to be proportional to the density of the air, and in bodies of similar figures, to be in the inverse ratio of their homologous dimensions, and of the density of the body. Call the density of the air D′, let m be a constant co-efficient, D the density of the body, and r the homologous dimension, which, in a spherical body is the radius, we have

$$gk^2=\frac{m\,D'}{r\,D};$$

whence we have for the value of k

$$k=\sqrt{\left(\frac{m\,D'}{gr\,D}\right)};$$

and for the constant velocity which cannot be exceeded,

$$v=\sqrt{\left(\frac{gr\,D}{m\,D'}\right)}. \tag{237}$$

All things else being equal, the maximum velocity will be proportioned to the square root of the density of the falling body; and hence, the denser the body, the longer it will continue to be accelerated, the greater will be the constant velocity acquired, and the shorter the time of its descent through a given distance.

So also, all things else being equal, the constant velocity will be proportioned to the square root of the radii of spherical bodies, and hence the larger the body of the same material, the greater will be its constant final velocity, and the less the time of its fall through a given distance.

241. The law thus ascertained in respect to bodies falling through the air, namely, that their acquired velocity can never exceed a certain limit, and finally becomes uniform, is true in all cases, where a body impelled by an accelerating force is retarded by another force, whose intensity increases in a higher ratio than the simple velocity. It is also true, as will be obvious, when the accelerating force decreases with the increase of the velocity, and the retarding force is constant.

It will be seen from (237) that if the density of the air vary, as is actually the case, the resistance must vary also; and that the co-efficient of the square of the velocity cannot be constant, as we have assumed in our hypothesis, but will be less in rare than in dense air.

To investigate the motion of a falling body, in such a manner as to include all the circumstances, it would be necessary then to take into account its figure and density; the variation in the intensity of gravity at different distances from the surface of the earth; the variation in the density of the atmosphere under different pressures and at different temperatures. The problem would therefore become extremely complex, even were the resistance of air of a given density to increase exactly, as assumed in our hypothesis, with the square of the velocity. It fortunately happens, however, that there are few or no cases in practical mechanics, in which a greater degree of accuracy in the determination of the motion of falling bodies is required, than is to be obtained from the original formulæ (231): and hence, even the investigation we have copied from Venturoli, and which may be found under another form in Poisson, is almost wholly a matter of mere curiosity.

When a body falls from rest under the action of the attraction of gravitation, as the direction of the force passes through its centre of gravity, it acquires no rotary motion. This is not necessarily the case when it is projected upwards, for the projectile force may be applied to a point other than its centre of gravity; the body will, in consequence, assume a rotary motion as well as one in a vertical direction. It therefore becomes necessary that we should consider the circumstances of motions of this character.

CHAPTER II.

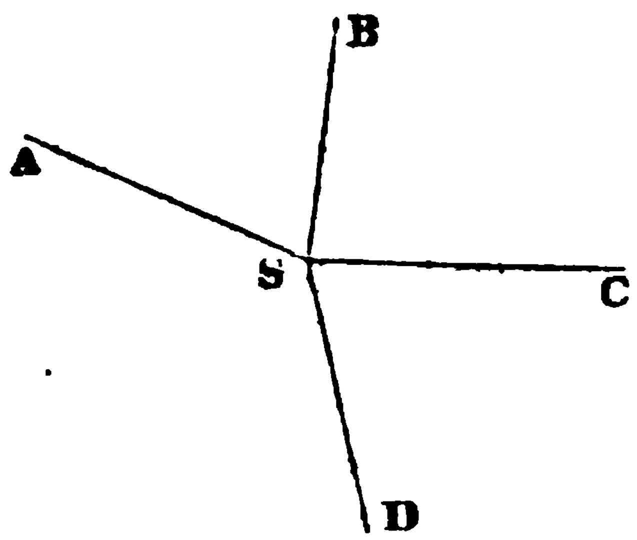
Of the Rotary Motion of Bodies.

242. It has been shown, § 82, that the measure of the moving force of a body is the product of its mass into its velocity. This is also called its quantity of motion, and is the measure of a force, that, if acting for an instant, and then abandoning the body to itself, would communicate to it the given velocity.

A solid body may be considered as a number of material points or particles of matter connected with each other in such a way as to form an invariable system. If each of these points be acted upon by an equal and parallel force, or if a single force, or the resultant of several, be applied to the centre of gravity, or in a line whose direction passes through the centre of gravity, the body will move in a straight line; and as the points will be in equilibrio around the centre of gravity, they will each proceed also in a straight line. But if the forces that act upon each point be not equal and parallel, or the resultant do not pass through the centre of gravity, each point will have a tendency to move in the direction and with the intensity of the force impressed. This tendency will be modified by the mutual connexion of the points; and hence, although each forms a part of a mass whose general velocity, if the forces cease to act, is constant, yet as each will have a different velocity, a rotary motion must ensue, around some point comprised within the system. It therefore becomes necessary as a preliminary to the general investigation of the conditions of the motion of solid bodies, to determine the laws of rotary motion.

Let us first assume that but a single force acts, and that the body must revolve around a fixed axis.

Let ABCD be a system of points lying in one plane, and invariably connected with each other, and with an axis passing through the point S, whence the distances to the points respectively are a, b, c, d; let A be the point to which the force is



applied, and v the velocity it would have were it not connected with the other points, u the velocity it has in consequence of its forming a part of the system. The quantity of motion lost by A, on this account, will be

$$A\,(v-u)\,;$$

and the moment of rotation of this force, in respect to S,

$$A\,u(v-u)\,.$$

As the whole system is invariably connected, the time of revolution of all the points will be the same, and their respective velocities will be to that of A, as their distances from S; hence their respective velocities will be

$$\frac{bu}{a},\quad \frac{cu}{a},\quad \frac{du}{a}.$$

The quantities of motion acquired by B, C, & D, will therefore be

$$\frac{B\,bu}{a},\quad \frac{C\,cu}{a},\quad \frac{D\,du}{u};$$

and their respective moments of rotation

$$\frac{B\,b^2u}{a},\quad \frac{C\,c^2u}{a},\quad \frac{D\,d^2u}{a}.$$

By the principle of D'Alembert, these several moments of rotation must be in equilibrio with each other, or

$$Aa\,(v-u)=\frac{Bb^2u+Cc^2u+Dd^2u}{a},$$

whence

$$u=\frac{A\,a^2v}{A\,a^2+B\,b^2+C\,c^2+D\,d^2}. \tag{238}$$

The moment of the system in respect to the point S, will be

$$(A\,a^2+B\,b^2+C\,c^2+D\,d^2)\,\frac{u}{a}; \tag{239}$$

each point in the system will therefore exert a force determined by multiplying it by the square of its distance from the axis, and by the velocity at A.

The sum of these products, extended to any number of points,

$$A\,a^2+B\,b^2+C\,c^2+D\,d^2+\&c. \tag{240}$$

is called the Moment of Inertia of the system, in respect to the fixed point.

The angular velocity is $\frac{u}{a}$, which is the value of the angle de-

scribed around the point S, in a second of time, in parts of the radius; and to determine it in portions of a circle, the quantity $\frac{u}{a}$ must be multiplied by 57°. 29578; the value of the arc that is equal to the radius.

For the value of the angular velocity we have from (238)

$$\frac{u}{a}=\frac{\mathrm{A}\,av}{\mathrm{A}\,a^2+\mathrm{B}\,b^2+\mathrm{C}\,c^2+\mathrm{D}\,d^2}. \tag{241}$$

These propositions are obviously true of a system composed of any number of points whatever, situated in the same place. They are also true of a system lying in different planes, the distances a, b, c, &c., being in this case the perpendiculars let fall from the several points upon the fixed axis.

243. In any system of points, or bodies, that are compelled to revolve around a fixed axis, there may be found a point in which if they were all collected, a given force applied at any distance from the axis will communicate the same angular velocity as if it were applied at the same distance from the axis to the system, in its original state. This point is called the Centre of Gyration.

To find the centre of gyration, in the same system that we have just considered.

Let x be the distance of the centre of gyration from the axis: the moment of inertia of the system, if united in this point, will be

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D})\,x^2;$$

and as the angular velocity $\frac{u}{a}$ is to be the same in both cases, this must be equal to the moment of inertia of the system in the original state, or

$$(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D})\,x^2=\mathrm{A}\,a^2+\mathrm{B}\,b^2+\mathrm{C}\,c^2+\mathrm{D}\,d^2,$$

whence

$$x=\sqrt{\left[\frac{\mathrm{A}\,a^2+\mathrm{B}\,b^2+\mathrm{C}\,c^2+\mathrm{D}\,d^2}{\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}}\right]}. \tag{242}$$

In order to apply this to the case of a solid body, the number of points must be supposed infinite: call each of them dm, and the variable distance from the axis, r, the formula will become

$$x=\sqrt{\left(\frac{\int r^2dm}{\int dm}\right)}. \tag{243}$$

244. There will be a point in the radius SA, to which if an obstacle be applied sufficient to stop the rotary motion of the system, there will be no motion communicated to the axis S. And if a resistance be there applied, the whole of the force of the system will be exerted to overcome it. This point is called the Centre of Percussion. In this point then, if all the matter in the

system were collected, the moment of rotation will be equal to that of the system in its original state.

To find the position of this point: let x be its distance from S: the moment of rotation of the system if collected in that point will be

$$(A+B+C+D+\&c.)x.\frac{u}{a};$$

which must by hypothesis be equal to

$$Aa^2+Bb^2+Cc^2+Dd^2+\&c.\frac{u}{a};$$

whence we obtain for the value of x in all cases

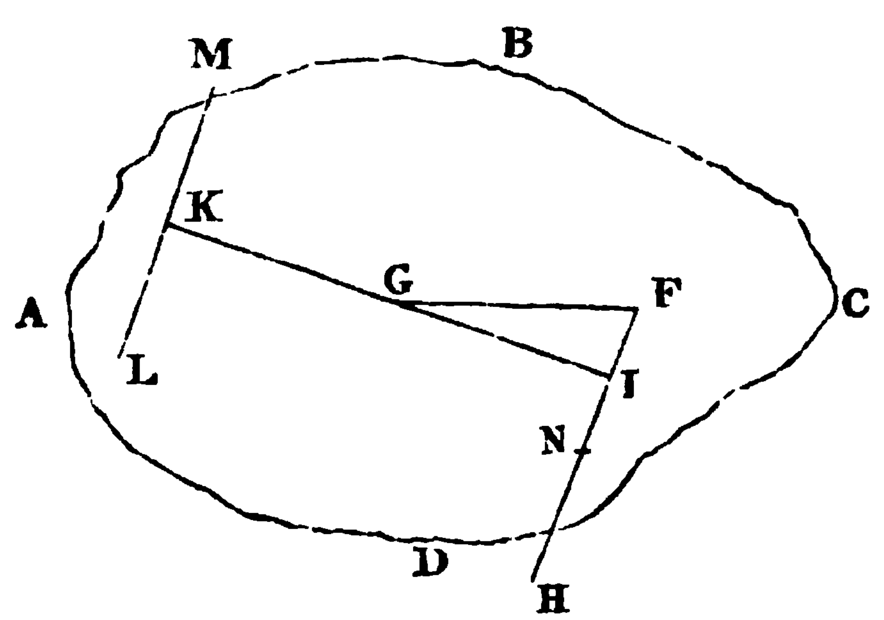
$$x=\frac{Aa^2+Bb^2+Cc^2+\&c.}{Aa+Bb+Cc+\&c.}; \tag{244}$$

and for the integral equation,

$$x=\frac{\int r^2dm}{\int rdm}. \tag{245}$$

245. We have seen that a force whose direction passes through the centre of gravity of a body, would give it a rectilineal direction only; while if the force do not pass through the centre of gravity, it must cause it to revolve. The case of its being compelled to revolve on a fixed axis has been examined. If, however, the body have no fixed point, it will not only acquire a rotary, but a progressive motion, or one of translation. In order to examine the manner in which these two different species of motions will take place:—

Let ABCD be a section of the body, passing through its centre of gravity, G, and the point F to which the force that produces the motion is applied; let FH represent this force in magnitude and direction; from the centre of gravity G, draw GI perpendicular to FH; and in GI produced on the other side of G, take GK equal to



GI. It will be evident that the condition of equilibrium of the

system will not be changed by applying two forces, KL & KM, to the point K, each of which is equal to the half of FH, provided they be parallel to FH, and act in opposite directions. The force FH may then be considered as the resultant of four forces, say its two halves FN, and HN, and the two assumed forces, KL and KM. Of these four forces, two, FN and KL, concur to produce rectilineal motion, in the direction, and with the intensity of their resultant; the direction of their resultant passes through the point G, is parallel to their direction, and equal in magnitude to the original force FH. The other two forces, HN and KM, will concur to produce a rotary motion, around an axis passing through the same point G; and this axis will be a normal to the plane ABCD: hence —

If a force be applied in any direction to a body which is free to move, it will cause its centre of gravity to describe a straight line, parallel to the direction of the force; and will communicate to the body a quantity of motion equal to its own intensity. The velocity may of course be found by dividing the force by the mass of the body.

If the force do not pass through the centre of gravity, it will, besides, communicate to the body a rotary motion around an axis passing through its centre of gravity; and this axis will be a normal to the plane, passing through the centre of gravity and the direction of the force. The angular velocity, and other circumstances of the rotary motion, may be computed as if the axis of rotation were fixed. For it will be at once seen, that if we were to apply to the body a force in a direction passing through the centre of gravity, equal in magnitude, and contrary in direction, to its motion of translation, we should destroy this motion altogether, but should not in any manner affect its motion of rotation.

When a body revolves upon an axis, every point will acquire a centrifugal force proportioned to its distance from the axis. Hence the axis of rotation will have a constant position, only when these centrifugal forces are in equilibrio around it. In all other cases, the axis of rotation must undergo a change. A homogeneous sphere may revolve permanently upon any one of its diameters. An ellipsoid of revolution may revolve permanently around the axis of the generating curve, or upon any one of its equatorial diameters; but upon no other line, for the several points that compose this solid will not be symmetrically situated in respect to any other. A homogeneous cylinder may revolve permanently upon its geometric axis, or upon any diameter of the circle that bisects the axis.

246. A further examination of the properties of revolving bodies, leads to a remarkable proposition; the investigation of which exceeds the limits within which our subject is restricted. It is as follows:

In any body whatever, however irregular, there are three axes of permanent rotation at right angles to each other, upon any one of which, if the body revolve, the centrifugal forces of its several points will be in equilibrio. These three axes, have also this remarkable property, that the moment of inertia in respect to them, is either a maximum or a minimum; that is to say, is greater or less than if the body revolved around any other line as an axis.

247. When a body has a double motion, of rotation and translation impressed upon it, the centre of gravity will, as has been seen, move forwards with uniform velocity; the other points in the body will move with velocities that are continually varying. Those on which the rotary and direct motions concur for an instant, and which are most distant from the axis of rotation, will move with the greatest velocity; and those in which these motions are opposed, will move with the least; and some of the points, most distant from the axis of rotation, will actually have a motion in a direction contrary to that of the centre of gravity.

248. There will also be a point in the system, in which at any instant, the progressive and rotary motions will exactly balance each other. This point is called the Centre of Spontaneous Rotation. The motion of the body for any instant of time, may be considered as a simple rotary motion around this centre. This variety in the rate at which the different points of a body move, when endued both with a motion of rotation and translation, has no effect when the body moves forward, without meeting with resisting forces; but when these act, it produces marked changes in the direction and circumstances of their motion. We shall have occasion to recur to these circumstances hereafter, in treating of motions in resisting media. For the present we shall confine ourselves to what happens when a body is rising or falling in the air, near the surface of the earth.

The resistance it meets with from that medium, being a function of the velocity, will act unequally upon opposite sides of the body, unless the rotation be around a vertical line; and this unequal resistance will produce a deviation from the original direction of the motion. Hence, a heavy body, although projected vertically upwards, rarely or never falls back upon the exact point whence it was projected.

There are various instances of the same kind to be met with, when the air or other resistances act upon a revolving body, and the deviation may become so considerable, as to bring a body projected horizontally, back to the point where the motion began. Thus, if a disk of metal, such as a coin, placed in a vertical position, upon a plane surface, be impelled by a force applied to either extremity of its horizontal diameter, it will acquire a rotary and a progres-

sive motion, the former being around its vertical diameter; the unequal action of the air upon the opposite sides of the vertical axis, concurring with the friction of the plane on which it rests, will cause it to describe a series of re-entering curves. A billiard ball placed on a plane surface, and impelled by a force which gives its lower side a motion of rotation contrary to the direction of its centre of gravity, will have its progressive motion destroyed by the friction, and will, afterwards, by virtue of the rotary motions it retains, roll back towards the place whence it originally set out. Cases of a similar nature are too frequent and familiar to need enumeration.

249. The general expression, (240), for the value of the moment of inertia, may be applied to particular cases by means of the calculus.

Call the moment of inertia S; let dm be an element of the figure whose moment is sought, and x the distance from the axis of rotation; then

$$S=\int x^2dm. \qquad (246)$$

To adapt this to individual instances:

(1). To find the moment of inertia of a straight line, revolving around an axis perpendicular to itself, and passing through one of its extremities:
then dm becomes dx, and

$$S=\int x^2dx;$$

integrating

$$S=\frac{x^3}{3};$$

and when $x=a$,

$$S=\frac{a^3}{3}. \qquad (247)$$

(2). To find the moment of inertia of the circumference of a circle, in respect to an axis passing through the centre, and perpendicular to the plane of the circle:

The element of the curve being ds, and its distance from the axis or radius, a,

$$S=\int a^2ds;$$

integrating

$$S=a^2s,$$

and when s becomes the whole circumference, or $2\pi a$,

$$S=2\pi a^3. \qquad (248)$$

(3). To find the moment of inertia of the circumference of a circle in respect to a diameter; let x and y be the ordinate and abscissa; the element $ds=\frac{adx}{y}$; its distance from the diameter $=y$, and

$$S=a\int ydx.$$

The integral of ydx, when s becomes the whole circumference, is equal to the area or to πa^2, therefore,

$$S=\pi a^3. \qquad (249)$$

(4). To find the moment of inertia of the area of a circle whose radius is a, in respect to an axis passing through its centre, and perpendicular to its plane:

Take an elementary ring whose radii are z, and $z+dz$, the area of this ring will be, $2\pi zdz$, and its moment of inertia, considering it as the circumference of a circle, will be $2\pi z^3dz$, by case (2); hence,

$$S=2\pi\int z^3dz.$$

Integrating

$$S=\tfrac{1}{2}\pi az^4;$$

and when $z=a$,

$$S=\tfrac{1}{2}\pi\tfrac{1}{2}\pi a^4. \qquad (250)$$

(5). In a similar manner it may be concluded, that the moment of inertia of the area of a circle in respect to one of its diameters, is

$$S=\tfrac{1}{4}\pi a^4. \qquad (251)$$

(6). To find the moment of inertia of a solid, formed by the revolution of a curve, in respect to the axis of rotation:

The figure being symmetric may be referred to no more than two co-ordinates, x and y; take for the element the solid contained between two circles, whose distances from the origin of the co-ordinates are respectively x, and $x+dx$, the moment of rotation of the element will be, by case (3),

$$\tfrac{1}{2}\pi y^4dx;$$

and that of the whole solid,

$$S=\tfrac{1}{2}\pi\int y^4dx. \qquad (252)$$

To apply this formula to the case of a cylinder, whose length is b, and the radius of whose base is a,

$$S=\tfrac{1}{2}\pi a^4b. \qquad (253)$$

In a cone, the radius of whose base is a, and whose altitude is b,

$$S=\tfrac{1}{10}\pi a^4b. \qquad (254)$$

In a hemisphere,

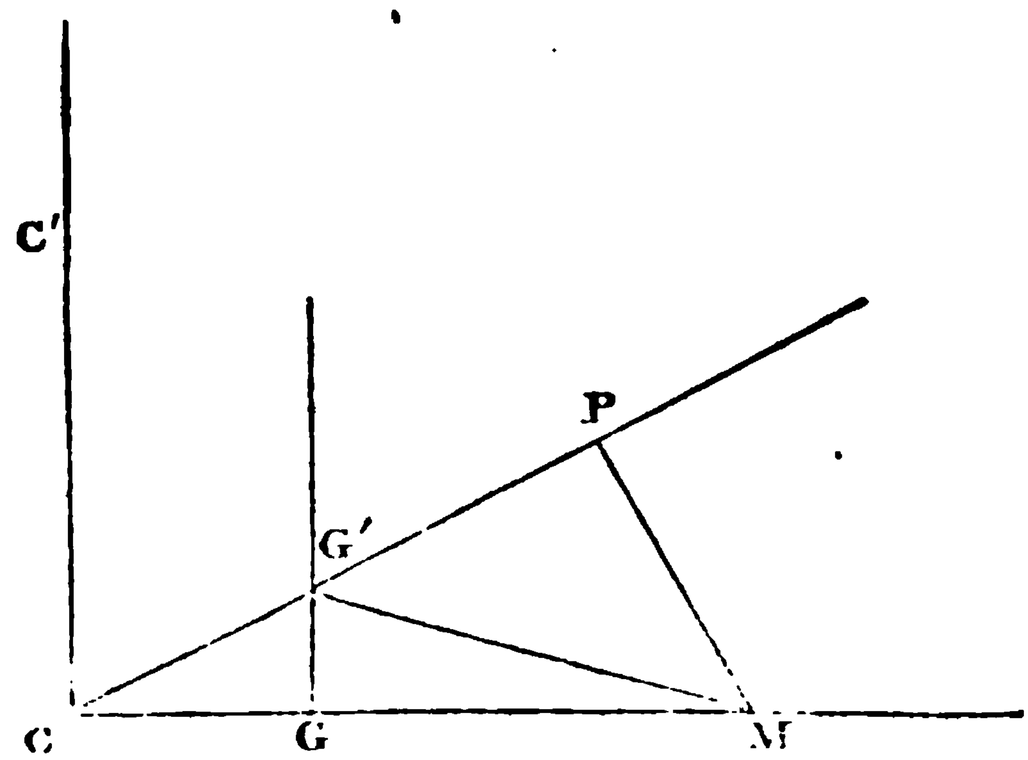
$$S=\tfrac{4}{15}\pi a^5. \qquad (255)$$

In a sphere,

$$S=\tfrac{8}{15}\pi a^5. \qquad (256)$$

250. When the moment of inertia of a body in respect to any axis is known, it is easy to find its moment of inertia in respect to any other axis, provided it be parallel to the first.

First let the given axis pass through the centre of gravity: then let GG′ be this axis, and CC′ another parallel to it, in respect to



which the moment of inertia is sought; let an element of the system be situated at M, and let its mass be dm: suppose a plane to pass through M, perpendicular to the two axes GG′, and CC′, and in this plane draw the lines MG′, MC, and join CG′; let fall MP from the point M, perpendicular on CG′. Let $MG=x$, $MC=y$, then

$$S=\int dmx^2, \qquad S'=\int dmy^2;$$

in the right angled triangles MCP, and MG′P,

$$MC^2=MP^2+CP^2$$

$$MP^2=MG'^2-G'P^2,$$

and

$$CP^2=(G'P+CG')^2,$$

whence

$$MC^2=MG'^2+2(CG'+G'P)+CG'^2,$$

and S′ will be equal to $\int dm$, multiplied by the second member of the above equation; but dm multiplied by the line GP, is the moment of inertia of the element dm, in respect to a plane passing through the centre of gravity G; and, therefore, $\int dm$ multiplied by the varying length of this line $=0$, by the property of the centre of parallel forces, § 25. The term of which it forms a part is, therefore, also $=0$, and disappears; hence, if we call the distance between the centre of gravity, and the point C, through which the second axis passes, k;

$$S'=\int dmx^2+\int dmk^2; \qquad (257)$$

and by substituting S, for its value, and integrating the second term, we obtain

$$S'=S+mk^2. \qquad (258)$$

The moment of inertia S″, in respect to any other parallel axis, whose distance from the centre of gravity is k', is

$$S'' = S + mk'^2; \tag{259}$$

and by substituting the value of S, obtained from the last equation,

$$S'' = S' + m(k'^2 - k^2). \tag{260}$$

251. This principle may be applied to the discovery of the moment of inertia, in cases different from those that have already been investigated. Thus:

(6). To find the moment of inertia of a circle, in respect to an axis parallel to a diameter: we have for its moment of inertia in respect to that diameter, (251), $\frac{1}{4}\pi a^4$:

Call the distance between the diameter and axis, k; m becomes the area of the circle, or πa^2, and

$$S = \tfrac{1}{4}\pi a^4 + \pi a^2 k. \tag{261}$$

(7). To find the moment of inertia of a solid of revolution, in respect to an axis passing through its vertex, and perpendicular to the axis of rotation x:

We may take for the element of the solid in this case, a circle whose radius is y, and whose distance from the axis is x. Its moment of inertia, in respect to the axis, will be, § 247,

$$\tfrac{1}{4}\pi y^4 dx;$$

and to the vertex,

$$\tfrac{3}{4}\pi y^4 dx;$$

and its mass is

$$\pi y^2 dx;$$

whence

$$S = \tfrac{1}{4}\pi \int y^4 dx + \pi \int x^2 y^2 dx. \tag{262}$$

If the axis pass through any other point, whose distance from the centre of gravity is r,

$$S = \tfrac{1}{4}\pi \int y^4 dx + \pi r^2 \int y^2 dx. \tag{263}$$

CHAPTER III.

Of the Motion of Projectiles.

252. It has been shown, § 238, that a body projected upwards from any point on the earth's surface, is retarded both by the action of the force of gravity, and the resistance of the air. If projected downwards from a point near the earth, the former would accelerate its motion with its whole intensity; the latter would still act to retard it. But if it be projected in any other direction than a vertical line, the resistance of the air continues to act according to the same law, while the force of gravity is now exerted in a direction inclined to that in which it would tend to move, under the action of the projectile force. Let us in the first place conceive the resistance of the air to be removed. To whatever point of the body the projectile force is applied, its centre of gravity, § 244, will acquire a motion, in a direction parallel to that of the force, with uniform velocity. The force of gravity, within the limits in which projectiles move, may be considered as constant, and although its directions are all converging, no important error can arise from considering it as acting always parallel to itself. The case, therefore, becomes the same as that we have considered in Book II. Chap. IV. Hence we may infer:

That if a body were projected from a point near the surface of the earth in a direction that is not a normal to that surface, and no other accelerating force were to act but that of gravity, it would describe a parabola, whose diameters are normals to the horizontal plane:

That it would have the greatest horizontal range, when projected at an angle of 45 degrees to the horizon:

That it would rise to the greatest height, the more nearly its original direction approaches to the vertical; and that the time of flight will also increase with the increase of the angle of elevation.

If we call h the height whence the body would have fallen to acquire the initial velocity v; d the horizontal distance to which the projectile is carried; i the angle of inclination; and y the force of gravity, we obtain from (72) and (61),

$$d=\frac{v^2 \sin. 2i}{g}, \qquad (264)$$

which is a maximum when

$$i=45^\circ;$$

in which case the equation becomes

$$d=\frac{v^2}{g}=\frac{v^2}{32}. \tag{265}$$

If we suppose the initial velocity to be 2000 feet per second,

$$d=125000 \text{ feet},$$

or nearly twenty four miles.

253. When projectiles move with but small velocity, the discrepancy between the parabolic theory, and what is found to occur in practice, is but small; but with increasing velocities, as the air's resistance increases in a higher ratio than the velocity, this discrepancy becomes very great. The general effect of a resisting medium may be thus investigated.*

Let gR represent the resistance of the medium, which will be exercised in a direction contrary to that of ds, the element of the curve at any given point. Refer the curve to two axes, one vertical, y, the other horizontal, x, both passing through the point of projection.

If we decompose the resistance, gR, into two forces parallel to the two axes, that parallel to x will be

$$g\mathrm{R}\,\frac{dx}{ds};$$

that parallel to y,

$$g\mathrm{R}\,\frac{dy}{ds};$$

and these being retarding forces, will have the negative sign. The first will express the whole disturbing force in the direction parallel to x; but in the direction parallel to y, the force of gravity acts also: hence the whole forces parallel to the two co-ordinates become

$$-g\mathrm{R}\,\frac{dx}{ds},$$

and

$$-g-g\mathrm{R}\,\frac{dy}{ds}.$$

From the general equations of curvilinear motion, § 52, we have

$$-g\mathrm{R}\,\frac{dx}{ds}=\frac{d^2x}{dt^2},$$

and

$$-g-g\mathrm{R}\,\frac{dy}{ds}=\frac{d^2y}{dt^2};$$

whence we have

$$-g\mathrm{R}\,\frac{dx}{ds}\,dt=\frac{d^2x}{dt},$$

*Venturoli, Vol. I. p. 95.

$$-g-gR\frac{dy}{dt}=\frac{d^2y}{dt}.$$

The velocity in the direction of x is constant by (236), therefore, by taking the differential of both, and substituting in the second the value of d^2t, we obtain

$$dsd^2t=gRdt^3;\quad d^2y=-gdt^2;$$

and taking the differential of the second, we have

$$d^3y=-2gdt\,d^2t;$$

eliminating dt and d^2t from this equation, we have for the equation of the curve,

$$2Rd^2y^2+dsd^3y=0.$$

Whence, if the law of the resistance were known, we might proceed to determine the nature of the curve.

Assuming that the resistance of the air varies with the square of the velocities, it has been attempted to ascertain the curve which a projectile would describe. Its differential equation has not, however, yielded to the integral calculus; and were the resistance to bear, as we hereafter shall see that it does, a more complex relation to the velocity, even the differential equation would become difficult to obtain. Although this curve does not satisfy the actual conditions of the resistance, it will still be a matter of interest to state what is known in respect to it, from its differential equation, and from a method of determining points through which it passes, that has been thence deduced. While the two branches of the parabola, measured from the horizontal plane, are similar and equal to each other, and have equal bases; the two branches of the curve that would be described under the action of a projectile force, the attraction of gravitation, and a resistance varying with the squares of the velocity, are unequal; the base of its ascending branch being longer than that of the descending branch.

In the parabola, the times of describing the two branches are equal; and the velocity of the projectile, when it again reaches the horizontal plane passing through the point of projection, is equal to the initial velocity. In the curve under consideration, the time of describing the ascending branch is less than that of describing the descending branch, and the final velocity is less than the initial velocity, and may become constant.

254. The most important application of our subject is to what are styled military projectiles. These are impelled by the explosion of gunpowder from instruments having cylindrical cavities, and which are, in general terms, called Pieces of Ordnance.

When gunpowder is inflamed, the whole is converted into an

elastic fluid, whose bulk is many times greater than that of the solid whence it is derived, and whose elastic force is enhanced by the high temperature at which it is generated.

By the experiments of Hutton, to which we shall presently refer, gunpowder would appear to expand itself with a velocity of 5000 feet per second, and to exert a force at least 2000 times as great as the pressure of the atmosphere. This deduction was obtained from its action upon balls, fired from pieces of ordnance. The weight of these is small compared with the force exerted by the gunpowder, and their motion is at first but little resisted; hence they are set in motion before the whole of the gunpowder is fired, and do not sustain its entire action. It is even well established by experience, that no inconsiderable quantity of the gunpowder is often blown, uninflamed, out of the gun, by the explosion of the rest.

255. Count Rumford made experiments in another manner. The gunpowder was in very small quantities, and was placed in a small vessel of wrought iron, without a vent. The ignition was accomplished, by heating this vessel red hot from without. In this way the escape of elastic fluid, that takes place in pieces of ordnance from the vent, was avoided. The resistance to be overcome was a weight of great amount, when compared with the quantity of gunpowder, being a brass battering cannon of the size called a twenty-four pounder; the cascable, or spherical knob that terminated the breech of this gun, was fitted to the vessel containing the gunpowder, by grinding, in such a manner as to make an air-tight joint. Experimenting with this apparatus, Rumford inferred that the force exerted by confined gunpowder, was equivalent to the pressure of 100000 atmospheres.

This would, at first sight, appear to exceed the limits of credibility; but there are cases in which gunpowder does actually exert a force far greater than it could, were its energy no more than was inferred by Hutton. As instances of this kind, may be cited what happens in the blasting of rocks, and in military mines, particularly in the effect produced by the latter, that is called the globe of compression. In this, besides throwing up a large conoid of earth, the gunpowder shakes the ground to a considerable distance around it, acting with sufficient energy to overturn and destroy walls of solid masonry. There being this great difference in the action of gunpowder, when it is exerted against a body that is easily set in motion, and when it is closely confined; it will be at once seen that great dangers may arise, when, from accident or intention, the projectile to be launched from a piece of ordnance is resisted in its motion. Thus, if the muzzle of a gun be inserted in water, if a portion of air be left

between a wad and the rest of the charge; if the projectile be of a hard material, and of such a shape that it may strike before it issues from the piece: in all these cases, the strength of the material of which the piece is formed may not be sufficient to resist the accumulation of force, and bursting may be the consequence. So, also, if the wad be of a cohesive material, such as tarred yarn; and particularly when it is so large as to enter the piece with difficulty, similar consequences may ensue. To the latter cause we may with certainty attribute the bursting of guns in the navy of the United States, and to the frequent loss of them in the proof. We have ourselves witnessed a case in the proof of guns, where the balls made their way through the sides of the piece, and large portions of the wad remained sticking to the bore in front of them.

256. The theory of projectiles, as has been seen, cannot be completed by the aid of mathematics, and it hence becomes necessary, in order that the practice of gunnery may become sure, that experiment be made upon a considerable variety of pieces of ordnance, with projectiles of various descriptions, at different angles of elevation, and with varying charges of gunpowder. The best general experiments of this nature have been made by Hutton and Robins. In addition, the artillery services of several European nations, and particularly that of France, are in possession of manuscript tables of the effects of their several descriptions of ordnance, derived from the experiments that are annually making at their schools of practice. The experiments of Hutton, Robins, and some of less extent made by Count Rumford, being alone accessible, we shall be compelled to confine ourselves to them: they are, however, sufficient for the foundation of a correct theory, on which a sound practice may be established.

257. The first point to be ascertained in experiments on military projectiles, is the initial velocity, which serves as the basis of all the other investigations. In the older experiments, this was done by pointing the piece vertically upwards, and observing the time that elapsed between the discharge and the return of the ball to the ground; half of this was assumed as the time of descent, and the velocity calculated by the formula, (231)

$$v=32\,t.$$

This method was found to give a velocity, that, if small, agreed tolerably well, in the ranges calculated from it, with the parabolic theory. But at the usual velocities of military projectiles, it was found to give results that varied very much from that theory.

258. Robins next invented an apparatus styled by him the Ballistic Pendulum. This is composed of a large mass of wood, suspended by a rod, from centres on which it is free to vibrate. The ball is fired against this, and the velocity communicated to it being measured, that of the ball may be easily determined. The whole of the ball's motion, provided it did not pass through the wooden mass, would be communicated to the latter; or rather the ball and it would go on with a common velocity, whence the quantity of motion could be determined, by multiplying the velocity by the joint mass of the ball and the pendulum. This product, divided by the mass of the ball, would give the velocity of the latter.

So far, the principle is simple: a difficulty, however, arises in the determination of the velocity of the pendulum; this does not go on with uniform motion, but rises in a circular arc, until the force of gravity checks its motion, and causes it again to return to the point whence it set out. But if the arc be known, the doctrine of the motion of pendulums, which will be explained hereafter, enables us to calculate the velocity that is destroyed in describing it.

Another difficulty arises from the fact that the pendulum will describe different arcs, according to the greater or less distance of the point the ball strikes, from the centre of motion; and it will also be obvious from § 243, that if the ball be not fired against the centre of percussion, a part of the force will be expended upon the axis on which the pendulum hangs. This difficulty was removed by Hutton, who investigated a theorem whence the velocity could be calculated, when the position of the centres of gravity and of percussion, and that of the point where the ball strikes, are known; his formula is as follows: viz.

$$v = 5.6727\, g\, c\, \frac{p+b}{b\, i\, r} \sqrt{o};$$

in which b is the weight of the ball,

p the weight of the pendulum,

g the distance between the centre of gravity and the point of suspension,

o the distance between the centre of percussion and the point of suspension,

i the distance from the point struck by the ball to the point of suspension,

c the chord of the arc described by the pendulum, and

r its radius.

The position of the centre of gravity was determined experimentally, by balancing the pendulum upon an edge, according to the principles of § 114.

The quantity o, was ascertained by suspending the pendulum

and finding the time of its vibration in very small arcs, whence the length can be calculated, as will be hereafter explained in treating of pendulums.

The chord c, was measured, by attaching to the pendulum a graduated tape, which was drawn out by the motion of the pendulum, from between two steel edges.

Hutton also determined the initial velocity of the ball from the recoil of the gun, by means of the principle of inertia § 39. By this it is apparent that the gunpowder must communicate to the gun as much motion as it is capable of giving to the ball, but in a contrary direction. The gun rendered heavier by additional weights, was suspended in the same manner as the pendulum, and its velocity of recoil ascertained in the manner that has been described in the last section.

Experimenting in these ways, it was found that the greatest initial velocity of a military projectile, does not much exceed two thousand feet per second.

259. As the elastic fluid generated by the firing of gunpowder, acts upon the ball during the whole time of its continuance in the piece, tending to expand with a velocity of 5000 feet per second, while the ball does not acquire a velocity much greater than 2000 feet: the latter must be accelerated during the whole of its continuance in the piece, provided its velocity does not become so great, that the resistance of the air, and the friction to which it is subjected, are sufficient to render its velocity constant. This acceleration is not, however, uniform, but becomes less and less as the ball moves forward, for the following reasons:

(1.) The elasticity of the fluid proceeding from the inflamed gunpowder, decreases with the increase of the space it occupies.

(2.) The elasticity depends in part upon the temperature, and this will decrease as the gas expands, and also by the conducting power of the piece.

(3.) Forces of this nature act with more intensity upon bodies at rest than upon bodies in motion.

(4.) The ball is resisted by the air, a retarding force that increases in a higher ratio than does the velocity of the ball, and which may finally render the latter constant. See § 239.

Thus, although an increase in the length of the bore does, in all cases that can occur in practice, increase the initial velocity, still it does not do so in the same ratio in which the length of the bore is increased.

The experiments of Hutton showed that this increase was in a ratio not as great as that of the square root of the length of the piece, but in a ratio greater than that of the cube root; and that if l be the length of the piece, the ratio is almost exactly

$$l^{\frac{2}{5}}. \qquad (266)$$

260. The velocities communicated to balls of equal weights, in pieces of the same length, by unequal quantities of powder, were as the square roots of the quantities of powder.

The velocities communicated to balls of different weights, in pieces of equal lengths, by unequal quantities of powder, were inversely as the square roots of the weights of the balls.

If p be the quantity of gunpowder, b the weight of the ball, and m a co-efficient, constant for all pieces of similar form,

$$v=m\sqrt{\frac{2p}{b}}. \qquad (267)$$

This co-efficient, m, may be safely taken at 1600 feet in most of the cases that occur in practice.

A remarkable consequence follows from the above formula: If the weight of the ball be increased, by using solid instead of hollow balls, or making them of denser substances, the velocity, all other things being equal, does not decrease more rapidly than in the inverse ratio of the square root of the weight; and hence a heavy ball will have a greater quantity of motion than a lighter one, projected from the same piece, with equal quantities of powder.

261. The resistance of the air was also carefully examined by Hutton; we have not space to enter into the detail of his experiments: it is sufficient to state, that at all velocities, from 300 to 1100 feet per second, he found that the resistances might be thus expressed:

$$r=mv^2+nv. \qquad (268)$$

In this formula, v is the velocity, and m and n constant co-efficients, which for balls of the diameter of 2 inches, are,

$$m=\ .00002665,$$

$$n=-.00388.$$

The resistance to balls, is found to vary with the squares of their diameters. The formula for any other diameter (a) will therefore be

$$r=(mv^2-nv)\ a^2, \qquad (269)$$

in which

$$m=.000007657,$$

$$n=.000175.$$

Calculations founded upon this formula, are found difficult in practice; and hence a co-efficient is chosen from the experiments, which will affect only the square of the velocity, at such velocities as are most frequently employed in practice. Call this co-efficient c, the whole resistance to the body expressed in lbs. will be

$$cv^2\ a^2,$$

which will be equal to the value of the part of the retarding force in § 239, represented by k^2, multiplied by the weight of the body, or

$$bk^2=cv^2a^2;$$

whence

$$gk^2=\frac{cgv^2a^2}{b}; \qquad (270)$$

and gk^2 is the measure of the retarding force.

Let w be the initial velocity, v the velocity remaining after moving through a distance, x; in order to find the value of x:

By the general formula of variable motion, (53)

$$f=gk^2=\frac{dv}{dt}=\frac{v\,dv}{ds};$$

and substituting x for s,

$$f=\frac{v\,dv}{dx};$$

whence

$$v\,dv=f\,dx;$$

substituting the value of f or gk^2, and considering that the force is a retarding one, we obtain

$$-v\,dv=g\;dx.\frac{ca^2v^2}{w};$$

and

$$dx=\frac{w}{gca^2}\cdot\frac{-dv}{v}.$$

Integrating, and considering that when $x=o$, $v=u$, we obtain

$$x=\frac{w}{gca^2}\cdot\text{hyp. log. }\frac{u}{v}. \qquad (271)$$

To give this a more convenient form, and to adapt it to the case of balls of cast iron:

The weight of a cubic inch of cast iron is 4.3 oz., hence

$$w=.5236a^3\times 4.3=2.25a^3$$

in ounces avoirdupois, g, the measure of the force of gravity, is 32 feet, and the co-efficient, c, suited to velocities of from 1200 to 1400 feet per second is .000007657; hence we have

$$\frac{w}{gca^2}=581\tfrac{1}{4}a;$$

multiplying this co-efficient by 2.30258, in order that we may substitute common for hyperbolic logarithms, we obtain

$$x=1338a.\text{ log. }\frac{u}{v}; \qquad (272)$$

from this we obtain for the values of u and v,

$$\left.\begin{array}{l}\log.\ u=\dfrac{x}{1338a}+\log.\ v,\\ \log.\ v=\log.\ u-\dfrac{x}{1338a}.\end{array}\right\}\qquad(273)$$

A still more convenient form may be given to this, by considering that if D be the density of the ball,

$$w=.5236a^3\mathrm{D};$$

which, taking atmospheric air as the unit, gives us

$$x=\frac{20\ \mathrm{D}a}{9}\cdot\log.\frac{u}{v};\qquad(274)$$

whence

$$\left.\begin{array}{l}\log.\ u=\dfrac{9x}{20\ \mathrm{D}a}+\log.\ v,\\ \log.\ v=\log.\ u-\dfrac{9x}{20\ \mathrm{D}a};\end{array}\right\}\qquad(275)$$

The calculation of the two last formulæ could be made more easy, if they had the form

$$\left.\begin{array}{l}\log.\ u=\log.\ v+\log.\ m\\ \log.\ v=\log.\ u-\log.\ m.\end{array}\right\}\qquad(276)$$

If we make

$$\frac{20\ \mathrm{D}a}{9}=c,$$

this condition may be satisfied by finding a constant number, by which if $\frac{x}{c}$ be multiplied, it will give a logarithm, which, within the usual limits of the range of military projectiles, does not differ much from

$$\frac{9x}{20\ \mathrm{D}a};$$

the fraction 0.43429448, is one that is best suited for this purpose; hence, in the preceding formulæ,

$$\log.\ m=\frac{x}{c}\times 0.43429448.\qquad(277)$$

To find the time employed in describing the space x, we have (53)

$$dt=\frac{dx}{v};$$

from which may be obtained, by substitution and integration, employing afterwards the quantities c and m, according to the principles just laid down,

$$t=\frac{c}{u}\ (m-1);\qquad(278)$$

if the initial velocity, and the time of flight be given, we have for the value of the space x from (277)

$$x=\log.\ m\ \frac{c}{0.434};$$

and for m, from the previous equation,

$$m=\frac{ut}{c}+1. \qquad (279)$$

In order to determine the angle of projection at which a projectile with a given velocity will strike the horizontal plane at a given distance; we must consider, that the projectile, as soon as it leaves the piece, is acted upon by the force of gravity, by which, if we abstract from the air's resistance, it will in a time, t, fall through a space, s, whose value is (231)

$$s=16\ t^2.$$

If we consider the space, x, which the ball passes through, as coinciding with the horizontal distance, (and this would be the case if the ball described a straight line) we have, for the tangent of the angle of elevation i,

$$\text{Tan. } i=\frac{16t^2}{x}; \qquad (280)$$

and substituting the value of t from (278)

$$\text{Tan. } i=\frac{16}{u^2x}\ c^2\ (m-1)^2; \qquad (281)$$

from which may be obtained, by substituting the value of m,

$$\text{Tan. } i=\frac{16}{u^2}\ \left(\frac{x^2}{c}+x\right); \qquad (282)$$

whence we have

$$u=\sqrt{\left[\frac{16}{\tan.\ i}\ \left(\frac{x^2}{c}+x\right)\right]}, \qquad (283)$$

$$x=c\sqrt{\left[\left(\frac{u^2\tan.\ i}{16c}+\tfrac{1}{4}\right)-\tfrac{1}{2}\right]}. \qquad (284)$$

262. To enable us to apply these formulæ to practice: Cast iron has a density, in terms of water, as the unit of 7.4, as determined by Hutton, who also assumes that the relation of the densities of water and air are as 1000 : $1\frac{2}{9}$; hence we have for the density of cast iron in terms of air, 6054.

We have next to ascertain the diameters of the more usual projectiles, in fractions of feet. These, as will be hereafter more particularly described, are, in the case of cannon, distinguished by their weights. Their denominations and dimensions, are as follows:

Ball of	Diameter in inches.	Diameter in feet.
42 lbs.	7.018	0.5848
32 lbs.	6.410	0.5342
24 lbs.	5.823	0.4852
18 lbs.	5.292	0.4410
12 lbs.	4.623	0.3852
9 lbs.	4.200	0.3500
6 lbs.	3.668	0.3057
4 lbs.	3.204	0.2670

BALLS OF.	DIAMETERS IN FEET.	DENSITIES.	LOGARITHMS OF $\frac{20\,Da}{9}=c.$	LOGARITHMS OF $\frac{1}{c}\times 0.43439448.$
42 lbs.	0.5848	6054.	3.8958373	5.7420470
32 lbs.	0.5342		3.8565338	5.7813505
24 lbs.	0.4852		3.8147507	5.8231336
18 lbs.	0,4410		3.7732685	5.8646158
12 lbs.	0.3852		3.7145162	5.9233681
9 lbs.	0.3500		3.6728979	5.9649864
6 lbs.	0.3057		3.6151253	6.0127590
4 lbs.	0.2670		3.5553412	6.0825431

The pieces of ordnance most frequently employed in the practice of artillery are, Cannon, Mortars, Howitzers, and Carronades. The diameters of the bore of all these different pieces are called their Calibres.

Cannon vary from twelve to twenty-four, or even more calibres in length; they are moveable upon two axles that are prolongations of the same cylinder, whose axis is a normal to the vertical plane passing through the axis of the piece, and which are called Trunnions; by these they are attached to a carriage, on which they may be moved from place to place. The form of the carriage admits them to be elevated or depressed, a few degrees above or below the horizontal plane. They are distinguished, according to their use, into Navy, Battering, Garrison, and Field Guns or Pieces.

The axis of the trunnions is usually in the vertical plane passing through their centre of gravity, but intersects it below that point.

Mortars are short pieces of ordnance whose trunnions are at their breech. By these they are adapted to beds. In the English service, they are permanently fixed to these beds, at an angle of 45° with the horizon. In the French, and in our land service, they are moveable upon their trunnions, so that they may be fired at any angle of elevation.

Howitzers have a form similar to that of mortars, but have their trunnions situated like those of cannon. They are mounted

on the same kind of carriages as cannon, but as they are shorter, are capable of greater angles of elevation.

Carronades are short cannon, used almost wholly in the naval service. They have no trunnions, but are connected with their carriages or slides by iron straps, passed through a cylindrical opening, made in a piece cast on their lower sides for the purpose.

All these pieces of ordnance are now cast solid, and the cylindrical cavity that contains the charge is cut out by a rotary motion; whence it is called the Bore.

The bore of mortars, howitzers, and carronades, is made of smaller diameter towards the breech; thus assuming the shape of two cylinders united by a portion of a spherical surface. The smaller part of the bore is of such length as to receive the maximum service charge of gunpowder, and is called the Chamber. Some cannon also have chambers, as have the better description of small arms. The formulæ that have been given above, are applicable to cannon, howitzers and carronades, but not to mortars, unless fired at small angles of elevation; a case that rarely occurs in practice; for it is only at small angles of elevation that the actual path of the projectiles approaches nearly to the distance, measured in a straight line.

The distance called *Point Blank*, in the English service, is estimated from the position of the piece to the point at which the curved path of the projectile intersects the horizontal plane on which the carriage is placed; the axis of the piece being also horizontal. The term, *de But en Blanc*, of the French, frequently translated point blank, supposes the line of sight to be directed in a horizontal line over the highest points of the breech and muzzle. As these lie on the surface of a cone, the line of sight is inclined to the axis of the piece, the latter of which is, in consequence, inclined upwards from the horizon; the path of the projectile being a curve, will cut the line of sight in rising, at no great distance from the mouth of the piece, and again descending, will cut it a second time, at a distance that will vary with the velocity, and the angle of inclination. The distance from the piece, to th[illegible] e the path of the projectile cuts the line of sight the s[illegible] is the distance *de But en Blanc*. To strike a poin[illegible] nce, the piece is aimed directly at the object, by mea[illegible] ut into the breech, and the swell of the muzzle; and t[illegible] ill be as accurate as any method in gunnery can well be, whether the point aimed at lies in the same horizontal plane with the piece, or be elevated or depressed in respect to that plane, within the limits of elevation and depression, that the carriage of the piece will admit.

In order to extend the advantages of the method of direct aim,

to objects more remote than the distance *de But en Blanc*, the French artillerists adapt a moveable sight, called the Hausse, to the breech of the piece; when this is raised, and the line of sight again directed to the object, the axis of the piece will be more elevated than before, and the horizontal range in consequence greater. When the object is nearer than the distance *de But en Blanc*, the original or natural line of sight must be directed to a point below the object.

The calculation of the angles of elevation corresponding to the natural line of sight, and to that given by known elevations of the *Hausse*, may be ascertained as follows:

The angle of elevation will be equal to the least angle of a right-angled plane triangle, of which one side is the difference between the radii at the breech, and at the muzzle, and the other the length of the piece: hence, if R and r, be the respective radii, and l the length,

$$\tan. i = \frac{R - r}{l} :$$

if now the *Hausse* be used, let H be the height to which it is elevated,

$$\tan. i = \frac{R + H - r}{l} .$$

The application of these principles and formulæ to practice, may be illustrated by the following

EXAMPLES.

(1). *A 24 pound ball is projected with a velocity of 1200 feet per second,—required the velocity it will have, after passing through a distance of 900 feet?*

The formulæ are,

$$\log. v = \log. u - \log. m, \qquad (276)$$

$$\log. m = \frac{x}{c} \times 0.4343 ; \qquad (277)$$

and

$$u = 1200 \text{ feet} ;$$
$$x = 900 \text{ feet} ;$$

log. $(u=1200,)$			3.0791812
log. $\frac{1}{c} \times 0.4343$ . . .	5.8231336		
log. $(x=900,)$	2.9542425		
log. $\left(\frac{x}{c} \times 0.4343 = \log. m\right)$	8.7773761	No.	0.0598930
log. $(v=1045)$			3.0192882

(2). *A 24-pound ball is fired at an object distant* 900 *feet, and will require there a velocity of* 1000 *feet to produce the proper effect,—required the initial velocity with which it should be projected?*

The formula is

$$\log. u = \log. v + \log. m; \tag{276}$$

The calculation of the preceding example gives

log. $m=$	0.0598930
log. ($v=1000$)	3.0000000
log. ($u=1147$ feet)	3.0598930

(3). *A 24-pound ball being projected with an initial velocity of* 1200 *feet per second,—required the time it will take to pass through the first* 900 *feet?*

The formula is

$$t=\frac{c}{u}\,(m-1); \tag{278}$$

we have, as before,

$$\log. m = 0.0598930,$$

and

$$m=1.147,$$
$$m-1=0.147;$$

log. $m-1$, ...	9.1673173
log. c, (from subsidiary table), ...	3.8147507
ar. co. log. ($u=1200$ feet,)	6.9208188
log. ($t=0.''799$,)	9.9028868

or nearly 8-tenths of a second.

4. *The elevation of the axis of a 24-pound gun, when pointed* de But en Blanc, *is* 1°.20′; *with what initial velocity, should the ball be projected to strike an object, distant* 2000 *feet?*

The formula is,

$$u=\sqrt{\left[\frac{16}{\tan. i}\left(\frac{x^2}{c}+x\right)\right]}; \tag{283}$$

log. ($x=2000$,)	3.3130300
	2
log. x^2, ...	6.6160600
ar. co. log. c, ...	6.1852493
log. $\left[\frac{x^2}{c}=632.86\right]$ Carried forward,	2.8013093

$\frac{x^2}{c}=632.86$ Brought forward,

$x=2000.00$

$\frac{x^2}{c}+x=2632.86$ log.	3.4204278
log. 16, ...	1.2041200
ar. co. tan. i, ...	1.6331055
	2)6.2576533
log. (u=1345.32)	3.1288266

(5). *At what angle of elevation should a 24-pound gun be fired, in order to strike a mark at a distance of* 2682 *feet, the initial velocity being* 1500 *feet per second?*

The formula is,

$$\tan. i=\frac{16}{u^2}\left(\frac{x^2}{c}+x\right) \qquad (282)$$

log. (x=2682 feet,)	3.4284588
	2
log. x^2	6.8569176
ar. co. log. c, ...	6.1852493
log. $\left(\frac{x^2}{c}=1102\right)$	3.0421679

$x=2682$

$\frac{x^2}{c}+x$ = 3784 log.	3.5779511
log. 16, ...	1.2041200
log. u^2=3.1760913×2 ar. co.	3.6478174
tan. (i=1°.32′.29″,)	8.4298885

(6) *What is the range* de But en Blanc, *of a* 24-*pound gun, whose natural elevation is* 1°.20′., *and initial velocity* 1400 *feet per second?*

The formula is,

$$x=c\left[\sqrt{\left(\frac{u^2 \tan. i}{16c}+\tfrac{1}{4}\right)}-\tfrac{1}{2}\right] \qquad (284)$$

log. (u=1400)	3.1461280
	2
log. u^2 Carried over,	6.2922560

log. u^2	Brought over,	6.2922560
tan. ($i=1^{\circ}.20$)		8.3668945
ar. co. log. 16		8.7958890
ar. co. log. c		6.1852493
log. 0.436797		9.6402798
add 0.25		
log. 0.686797		9.8368285
which divided by 2, gives		9.9684142
whose number is 0.931996		
deduct 0.500000		
log. 0.431996		9.6354798
log. c		3.8147507
log. 2819.88		3.4502305

The distance *de but en blanc* is therefore nearly 2820 feet.

The foregoing tables are not applicable to the case of projectiles, fired at angles exceeding four or five degrees, in consequence of the error which arises from taking the horizontal distance as equal to the actual path, and from considering the tangent of the angle of elevation to be equal to $\frac{16t^2}{x}$. Both of which assumptions are obviously far different from the truth, when the path acquires any sensible curvature. The approximations that have been made to the determination of the problem of the motion of shells, and other projectiles, fired at great angles of elevation, may be seen in Hutton's tracts. In actual service, tables are used, calculated for every particular species of ordnance; and hence, it is unnecessary to attempt giving any general rules.

263. The value which we have taken for the co-efficient of v^2, in the formulæ, is that which corresponds to initial velocities of 12 to 1500 per second. Greater velocities may be given, as has been stated, to military projectiles; say upwards of 2000 feet per second. To give these velocities, is, however, attended with a much increased expenditure of gunpowder; for the resistances increase, afterwards, in much higher ratio. The cause of this is, that the projectile, in passing through the atmosphere, forms a wave of condensed air in its front, while the air is rarefied behind it; it is hence resisted by a medium of greater density, while it derives little or no support from behind. At a ve-

locity of 1342 feet, the air will no longer follow it, and a vacuum will be left behind it: hence, any initial velocity, however great, will be speedily reduced to that limit, and will require to produce it a greater charge of gunpowder than would be consistent with the formula (267)

$$v=1600\sqrt{\frac{2p}{b}}.$$

A velocity of 1300 feet per second is given by a charge of one third of the weight of the ball; and this is the greatest charge that ought to be admitted in service, except in one particular case, to which we shall hereafter refer. This is the regulation charge for cannon in the British and American navies; and although the velocity it gives is often greater than is advisable, for reasons that will be presently stated, still it would be inexpedient to reduce it farther, in consequence of the injury that gunpowder sustains from the moist air, in contact with the ocean. In the land service, on the other hand, where this deterioration does not take place to such an extent, the service charge, except in the case that has been referred to, need never exceed $\frac{1}{4}$th or $\frac{1}{5}$th of the weight of the ball.

264. As balls for the larger species of ordnance are made of a hard material, cast-iron, they cannot fit the bore of the piece exactly, without endangering its bursting. Hence there is a difference between the calibre of the piece, and the diameter of the ball, which is called the windage. The more perfect the workmanship of the bore, and the more accurately the balls are cast, the less may be the windage. It has, therefore, been considerably diminished with the improvement of the mechanic arts. In different services, the practice, in this respect, has been different. In some, the windage has been made to bear a certain proportion to the diameter of the ball; in others, it is a constant quantity in all cavities. Were the principal dangers to be apprehended, a want of sphericity in the balls, or of regularity in the bore, the former would be the correct method; but in the present state of the arts in our country, the chief risk will arise from an increase in the diameter of the ball by oxidation, and this may be met by a windage constant in all calibres. Sir Howard Douglas has proposed that it be reduced in the British naval service to little more than $\frac{1}{8}$th of an inch, from 0.33 in. in the 42 pound gun, and 0.22 in. in the 12. With this reduced windage, he infers that an equal velocity may be given with $\frac{5}{6}$ths of the powder now used. That an increased velocity will be a consequence of the diminution of the windage will be obvious, when it is considered that the ball moves from rest to a velocity that never exceeds $\frac{5}{6}$ths of that with which the elastic fluid tends to expand itself. Hence a portion of the gas will escape without acting; and this will be deter

mined by the size of the eccentric ring, formed by the circular sections of the bore and the ball.

265. In consequence of the windage, the ball will rest on the bottom of the bore, or will strike against it as it passes out, even when supported by a wooden seat or bottom, in such manner that its axis coincides with the axis of the piece. In either of these cases, and one or other must occur, the ball will acquire a rotary motion around an axis that is not parallel to the axis of the piece. It will thus happen, as will be easily seen from § 241, that the ball will deviate, not only from a parabolic path, but from a plane curve; that it will, according to the direction of the axis around which the rotary motion takes place, be deflected to the right or to the left of the vertical plane, passing through the axis of the piece; and that it may rise above, or fall below the curve that would be described by a body not revolving. In striking against the bore, it may be reflected and thrown to the opposite side; and this may occur more than once before it leaves the mouth of the piece; this will cause a change in the initial direction, and concur in giving a rotary motion, by the combination of which, very considerable deviations may occur. These deviations may also be affected by irregularities, the ball passing from one side to the other of the same vertical plane. Such deviations may arise from cavities on the surface of the balls, or from their not being homogeneous within.

Inequalities in the bore may also contribute to cause deviations; and hence, particular guns will always throw their shot to the right, and others to the left of the line of sight.

The same weight even of the same parcel of gunpowder, will often produce different initial velocities; and different parcels often differ in strength. From all these circumstances, the practice of gunnery is attended with a great deal of uncertainty; and even the best theory is no more than a guide, and does not give results that are to be implicitly relied upon.

The deviations from the vertical plane, of which we have just spoken, will be enhanced by increasing the velocity; and hence it is, that charges less than those generally employed, will be most advantageous in many cases. For if a ball, after passing through a given space, retains sufficient velocity to do the injury it is intended to effect, it will strike the mark with more certainty, if discharged with a less initial velocity; and the required range will be better attained by an increased elevation, than by an increased charge. In all cases, greater uncertainty in the attainment of the object aimed at, is produced by the use of too great a charge of gunpowder.

266. The deviation of projectiles from the vertical plane, might be obviated by giving them a rotary motion around an axis co-

incidinig with the axis of the piece. This has, however, been found impracticable in the larger species of ordnance. In small arms, it may be effected by what is called rifling the barrel. This consists in cutting in the metal that surrounds the bore, shallow grooves, extending from the muzzle to the lodgment of the ball; these are spiral, making about one revolution, more or less, according to the length of the barrel, around the bore. The ball being of a soft metal, lead, is cast a little larger than the bore, and is, in loading, forced in; it is, therefore, cut by the hard sides of the bore into a form adapting itself to the spiral grooves or rifles; and when the piece is discharged, it derives from them a rotary motion around the axis of the piece. As there will be no windage, a less proportionate charge will give an appropriate velocity than in any other species of ordnance, and the aim is far more sure and certain.

267. It has been seen that the initial velocity, although increased by increasing the length of the piece, increases in a much less ratio, or with a power of the length between its square and cube root, which may be represented by $\frac{2}{5}$.

It has also been seen that great velocities are not attended with proportionate ranges, and cause uncertainty in the aim. It may hence be concluded that neither very great lengths in the bore, nor large charges of gunpowder, are ever necessary. With small charges, the metal of the piece is less strained than with large, and thus not only may the length, but the thickness of the piece be reduced. The results of the experiments of Robins and Hutton, have led to the lessening of the size and weight of most of the pieces of ordnance. A great and sudden improvement was, in consequence, made in the artillery services of Europe, about the commencement of the wars of the French revolution. No field-piece has now a bore of more than 18 calibres in length, which is, or was, lately, the regulation in the French service. In the English service, the regulation length is fourteen calibres, while in the American, during the late war, it was reduced to twelve, and the pieces weighed no more than 1cwt. to each pound of ball. These were found to be sufficient for all purposes of the service. An unwise policy has lately led to the alteration of the model, by giving the bore the proportions of the French pieces, yet without increasing the weight; it has, however, been found, that pieces of the new model, even after standing proof, have burst in the schools of practice.

In cannon other than field pieces, a reduction of the length is not always practicable: thus in battering guns, a certain length in front of the trunnions is absolutely necessary, for they are, generally speaking, fired from embrasures of earth, which would be injured by the gas expanding in every direction from the

mouth of a short gun. The battering guns of the French service are, therefore, made of the form of two frusta of cones, united at the trunnions; that nearest the breech diminishing more rapidly than that towards the muzzle. We shall see that this form, although well adapted for this object, is not as strong as one formed on another principle. The three calibres of French battering guns have all, for the same reason, equal lengths. Navy guns should also project to a certain distance beyond the side of the vessel; and the same reasons apply to garrison as to battering guns. It is, besides, convenient to have them of the same model, that the same pieces may be used for either purpose. In the American service, it may be here stated, instead of the three calibres used by the French for battering guns, the 24, 16, and 12 pounders, there is but one, the 18 pounder.

In field pieces, there is no objection to shortening the smaller pieces, and hence this class of guns, of each different nation, have bores of a constant number of calibres in length.

Cannon were formerly made of an alloy of copper and tin, which is generally, but improperly, called brass. It had the advantage of a great degree of tenacity, but was objectionable from its great density, and high cost. It was also liable, in rapid service, to soften and bend. Now that the charge has been reduced, cast-iron, although less tenacious than the brass, has been substituted for it, in the ship, garrison, and battering guns of European nations. But it has not been found practicable, generally speaking, to use cast-iron in the lighter species of ordnance, (field pieces); for although it is a general rule that small vessels, of similar dimensions and material, will bear a greater strain than large ones, still there is a circumstance in the casting of iron that more than counteracts this. It is found that articles made from the same cast-iron, and drawn from the same charge of a furnace, will be weaker in proportion as they are more rapidly cooled; and, therefore, the small masses of field pieces, cooling most rapidly, are weaker than the guns of other descriptions. But the iron made from the ores of Sweden and the United States, with charcoal, is of such excellent quality, that field pieces may be safely made of it, weighing even less than brass guns of equal calibre and length.

268. In respect to the shape of cannon, it will be at once seen that the part in which the charge of powder is situated, must bear the greatest strain; and it seems probable, although it would be difficult to reduce it to the test of mathematical analysis, that the greatest effort is exerted by the expanding gas, at the point where the ball is lodged. In all the cases of burst guns that we have examined, the breech was entire, a fact that seems to corroborate

this inference. Rumford has, therefore, proposed to make the thickness of metal greatest at this point, and has planned a gun of beautiful proportions, swelling in a curve from the breech to the point assumed for the lodgment of the ball, and again contracting in a curve to the projection of the muzzle. It is, however, impossible to assign the exact point at which the ball will be lodged, in consequence of the difference in the space occupied by different charges of gunpowder. Hence the form given to the American navy 32-pounder, and battering 18, is to be preferred; this is cylindric from the base ring to the trunnions, and conical thence to the swell of the muzzle. Our navy 42-pounder, which is formed on the principles of the French battering gun, having a great weight in the breech, and being comparatively thin at the lodgment of the ball, is much weaker under an equal weight, than if it were formed upon the same principles as the 32-pounder.

269. The next circumstance to be considered, is the manner, in which military projectiles penetrate the obstacles they encounter. The resistance of any homogeneous body may be considered as a uniform retarding force: hence, if we call this force f, the velocity with which the ball strikes the obstacle v, and the space to which it penetrates before it loses its whole motion s, we have from (61b)

$$s=\frac{v^2}{2f} \qquad f=\frac{v^2}{2s}.$$

The penetration of bodies of the same size, figure, and weight, will, therefore, be proportioned directly to the squares of their velocities, and inversely to the strength of the substance into which they enter.

If the bodies be balls of different diameters and densities, their own moving force, (their velocities being equal,) will be proportioned to their densities, and the cubes of their diameters; for it is as their masses; the body into which they enter, will resist with a force proportioned to the section of the ball, or to the square of its diameter; and, therefore, the depth penetrated, will be as the quotient of these quantities, or directly as the density and the diameter of the ball.

Introducing the cases of different velocities and different kinds of obstacles, we have for the general rule: that balls penetrate into obstacles to depths that are proportioned directly to their densities, their diameters, and the squares of their velocities; and inversely, to the resisting force of the obstacle.

If the same ball strike against the same obstacle, with different velocities, the depths are proportioned, as has been shown, to the squares of the velocities. The same rule will apply to the depth

that remains to be penetrated after a portion of the velocity has been lost, and the remaining velocities will be proportioned to the square roots of the remaining depths. Let us suppose the whole depth to be divided into units of length; the ratio between the original velocity and that remaining after penetrating any number of units, n will be

$$\frac{\sqrt{(s-n)}}{\sqrt{s}},$$

and the velocity lost will be

$$\frac{\sqrt{s}-\sqrt{(s-n)}}{\sqrt{s}}\,v.$$

The velocities lost in passing through the several units of space, will form a series of the following terms:

$$\frac{\sqrt{s}-\sqrt{(s-1)}}{\sqrt{s}}\,v,\quad \frac{\sqrt{s}-\sqrt{(s-2)}}{\sqrt{s}}\,v,\quad \frac{\sqrt{s}-\sqrt{(s-3)}}{\sqrt{s}}\,v,$$

$$\frac{\sqrt{s}-\sqrt{(s-4)}}{\sqrt{s}}\,v,\ \&c.$$

And it will be at once seen that this series is one that rapidly increases. Hence the velocity lost, and consequently the quantity of motion communicated, in passing through a given depth of any obstacle, is much greater at small velocities than it is at great.

It will therefore happen that balls passing with great velocities through thin obstacles, may do so without communicating any perceptible motion. When balls pass with great velocities through a substance that is elastic, the hole is smooth, and is often of less diameter than the ball; but if the velocity be small, the obstacle will be torn and rent.

The prodigious effect of modern military engines in destroying the walls of fortresses, depends upon the principle that the penetration is proportioned to the squares of the velocities. The actual quantity of motion in the battering ram of the ancients, might be made equal to that of a 24-pound ball; but the former produced no other effect than that of agitation, by which, however, a wall might be destroyed, after a long series of blows. When cannon are used for destroying walls, or as it is technically called, battering in breach, the shot of all the guns in the battery are first directed so as to cut, by their penetration, a horizontal groove in the lowest part of the wall that can be seen from their position. They are next directed so as to cut two vertical grooves at each end of the horizontal one; and thus a quadrangular portion is separated from the rest of the wall. The cannon are then fired in vollies against this portion, until the agitation it receives from the shock is sufficient to cause it to fall. In the latter part of the operation, cannon have no advantage over the battering ram, but

the former part of it, by which it is in fact rendered most efficient, could not be performed by the ram.

The effect being proportioned to the square of the velocity, this is the case to which reference was made in § 261, where the charge may exceed with advantage $\frac{1}{3}$d of the weight of the ball. It is usually made $\frac{2}{3}$ds. The batteries in breach are also established, as near as possible, to the wall to be destroyed, in order that the velocity may be but little diminished by the air's resistance.

In other cases, great velocities may be injurious: thus, if a ball pass through the obstacle, it will communicate less motion than if it lodge; and the greater the velocity, the less will be the injury done. For this reason, in close naval engagements, great velocities are less destructive of the enemy's vessel than smaller ones.

Upon this principle is founded the introduction of the carronade into the naval service. This species of ordnance is short; and being loaded with no more gunpowder than $\frac{1}{8}$th of the weight of the ball, may have but little thickness of metal. It may, therefore, be used in the place of long guns of much smaller calibres, while the effect of its projectiles, in close action, is greater than that of a ball of equal weight from a long gun.

270. When projectiles strike against a hard substance, in an oblique direction, they are reflected according to laws that will be hereafter examined. So, also, on the surface of water, balls impinging at small angles, rise again, and perform a second curved path; and this may be repeated several times. The resistance of the water will be proportioned to that component of the moving force of the ball, whose direction is a normal to the surface of the water: when this becomes less than the gravity of the ball, it will no longer rise. This motion, in successive bounds over the surface of ground or water, is called the *ricochet*. It has been applied to great advantage in the attack of fortified places; and gives to guns placed upon the shore, in proper positions, great advantages over those opposed to them in ships. In the attack of fortified places, the first or more distant batteries are no longer placed in front of the part to be attacked, but in the prolongation of its faces, and opposite to the returning sides of the fortress. The guns in these batteries are fired at elevations of 4° or 5°, with charges of gunpowder that enable the balls in the descending part of their path, just to raze the opposing parapet: they, therefore, bound along, parallel to the direction of the front to be attacked, dismount the guns, and destroy the defenders.

Under the fire of these ricochet batteries, approaches are made to points sufficiently near for the erection of batteries in breach;

by these the walls are destroyed. It is in consequence of this method, which was invented by Vauban, that the means of the attack of fortresses have become superior to those of defence, and that the time of the resistance of a fortress can be calculated with almost mathematical precision.

When balls are fired from shipping, and strike the water, they never rise in any of their bounds as high as the point whence they are projected. Hence, in firing from a ship at an object on the land that is higher than the deck on which the gun is placed, there is no other chance of striking it but by a direct aim; while, if a ball be fired from the land, it will strike the vessel, if it be within the limit of its recochets; and if the height of the battery be not so great as to permit the balls to rise above the vessel. It thus happens that two or three heavy guns, placed upon the land in a proper position, are more than a match for the heaviest ship, while towers, and walls with embrasures, may be destroyed by the superiority in the quantity of guns that can be arranged, in a given space, on shipboard.

CHAPTER IV.

Theory of the Pendulum.

271. If a gravitating body be suspended by a string or rod from a fixed point, it will hang in a vertical position; but if it be raised from that position laterally, the string or rod remaining inflexible, and then permitted to escape, it will, under the joint action of gravity, and the tension of the string or rod, descend in an arc of a circle of which the point of suspension will be the centre. When it reaches the vertical position, it will, § 58 have acquired a velocity equal to that acquired by falling vertically through the versed sine of the arc, and which would tend to carry it forwards with uniform velocity in a horizontal direction. The tension of the string will cause it to continue to move in the circle of which the point of suspension is the centre; it will, therefore, after passing the vertical line, rise in a circular arc, until its whole velocity be destroyed, which, if no other force but gravity act, will be, when it reaches a height on the opposite side of the vertical, equal to that whence it at first fell. On reaching this point, it will again descend, and passing the vertical, rise to the point whence it originally set out. From this it will a second time descend, and would thus continue to vibrate for ever. But it will be seen that two forces act to retard the motion, namely, the resistance of the atmosphere, and friction around the point of suspension. By virtue of these, the circular arcs described are gradually lessened, until a state of rest be reached; the suspending string or rod becoming stationary in a vertical position.

A body thus suspended and caused to vibrate, is called a Pendulum.

The passage from its highest position on one side of the vertical, until it reach the greatest height on the opposite side, is called an Oscillation.

272. In order to study the theory of this species of motion, we imagine to ourselves a gravitating point, suspended by means of an inflexible line, devoid of gravity. The case then becomes that of § 60, a gravitating point, compelled to move upon a circular surface, by an accelerating force whose direction is always parallel to itself.

The time t, of descent in a small circular arc, of which a is the radius, and h the versed sine, is by (84)

$$t=\frac{\pi}{2}\sqrt{\frac{a}{g}} \times \left(1+\frac{h}{8a}\right);$$

hence the whole time of an oscillation, T, becomes, substituting l the length of the pendulum for a,

$$T=\pi\sqrt{\frac{l}{g}} \times \left(1+\frac{h}{8l}\right); \tag{285}$$

and finally, in evanescent arcs,

$$T=\pi\sqrt{\frac{l}{g}}; \tag{286}$$

whence we obtain for the values of l and g,

$$\left.\begin{aligned} l&=g\,\frac{T^2}{\pi^2}, \\ g&=\frac{\pi^2 l}{T^2}; \end{aligned}\right\} \tag{287}$$

and when $T=1''$,

$$g=l\pi^2. \tag{288}$$

To compare the time of the oscillation of a pendulum in a very small arc, with that of the descent of a heavy body: Let the distance the body falls be half the length of the pendulum, and we have by (61)

$$t=\sqrt{\frac{l}{g}};$$

which compared with (286) gives

$$T : t :: \pi : 1. \tag{289}$$

When the respective times in which two pendulums vibrate, or their numbers of oscillations, in equal times, are known, and the length of one of them has been determined, that of the other may be calculated—

Let l and l' be their respective lengths;
T and T′ their times of oscillating;
N and N′ the numbers of vibrations in equal times, then
$T : T' :: N' : N$;

for the number of oscillations is inversely as the times.

We have for the value of the times (286)

$$T=\pi\sqrt{\frac{l}{g}}, \quad T'=\pi\sqrt{\frac{l'}{g}};$$

whence, $\quad T : T' :: \sqrt{l} : \sqrt{l'}$,

and $\quad N : N' :: \sqrt{l'} : \sqrt{l}$;

therefore, when the respective times are known, and l' is given,

$$l=\frac{T^2 l'}{T'^2};$$

and when N and N′ are known, and l' given,

$$l=\frac{N'^2 l'}{N^2}. \tag{290}$$

From these formulæ it may be concluded:

(1.) That, the force of gravity remaining constant, the times of the vibrations of pendulums of unequal lengths are respectively as the square roots of their lengths, [and inversely as the square roots of the numbers of their vibrations in a given time.]

(2.) That, the force of gravity still remaining constant, the lengths of different pendulums are directly as the squares of the times of their oscillations, and inversely as the squares of the numbers of their oscillations in a given time.

(3.) That the time of the oscillation of a pendulum, is to the time that a heavy body would fall freely by the force of gravity, through half its length, as the circumference of a circle is to its diameter.

(4.) In different positions, the intensity of the force of gravity may be measured by the length of the pendulum that vibrates in equal times at the different places; and the intensity of gravitation is always directly proportioned to the length of the pendulum that beats seconds at the place.

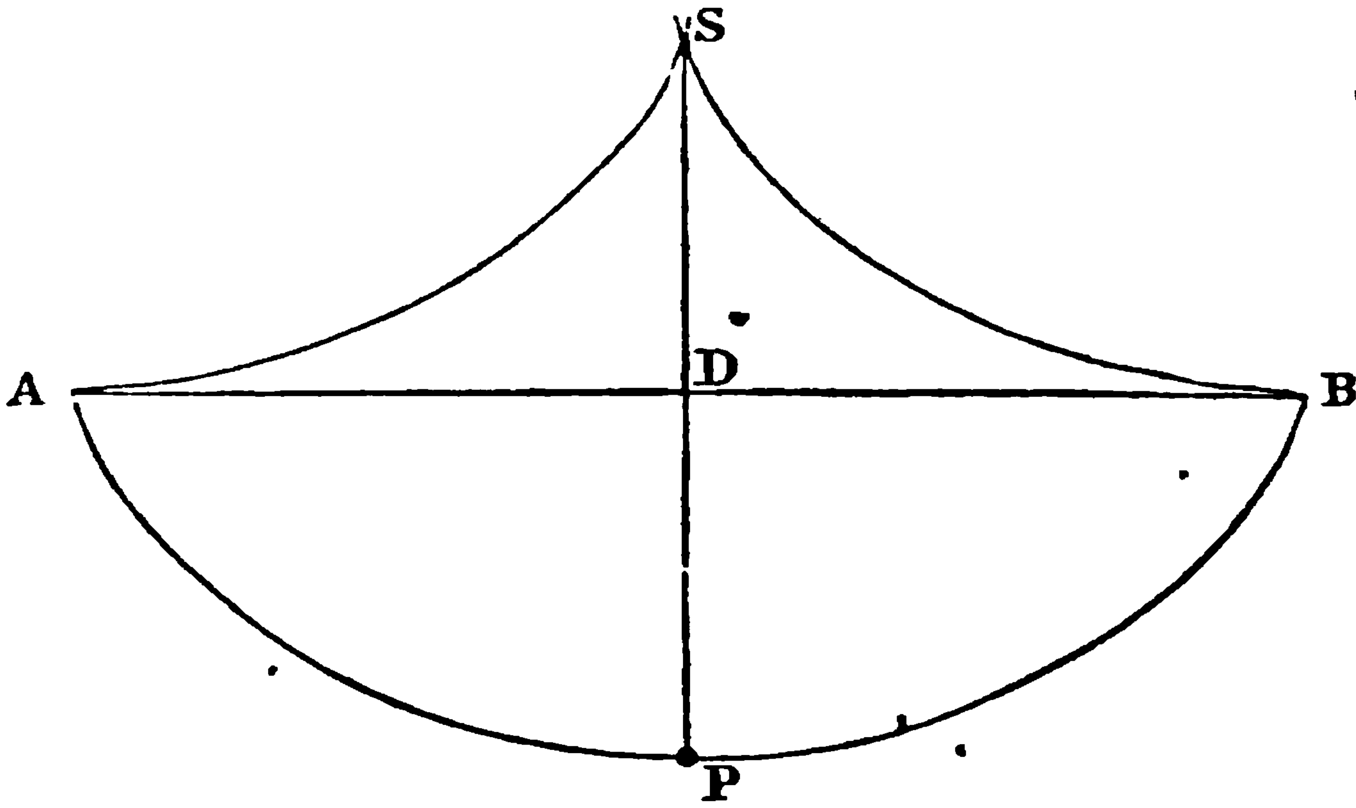
273. If the pendulum vibrate in a cycloid, the formula

$$T=\pi\sqrt{\frac{l}{g}},$$

becomes true, § 59, whatever be the amplitude of the arc: all the above propositions are, therefore, absolutely true of pendulums vibrating in cycloidal arcs.

It has been attempted to make pendulums vibrate in cycloidal arcs, upon the following principles, which are theoretically true; although, as we shall hereafter see, they are not susceptible of being applied to any practical purpose. This attempt has been founded on the following principles.

It is a property of the cycloid, that its evolute is an equal cycloid, placed in an inverted position. Hence, if the length of the pendulum SP be bisected, and on the line ADB, drawn



through the point of bisection, perpendicular to SP, a cycloid be constructed, the diameter of whose generating circle is DP; and if two half cycloids be constructed tangents to the lines SD, and ADB, the latter will be the evolutes of the corresponding semicycloids. If then a pendulum be suspended from S, by a flexible rod, of the length SP, it will, when moved from its vertical position and permitted to oscillate, apply itself to the two curves SA, SB; and it is evident, that during these oscillations, it will describe the curve APB, or a part of it.

When the circular arc does not exceed 4 or 5 degrees, the relation of the time of oscillation in it to that in a cycloidal arc is given by the formulæ (285) and (286), or is as

$$1 : 1+\frac{1}{8}h : \qquad (291)$$

when the arc exceeds that amount, it becomes necessary to introduce more terms of the series in (84).

From this series, the excess of the times of oscillation in circular arcs over those in a cycloid, has been calculated, as follows, viz.

In an arc of 30° on each side of the vertical . . .	0.01675
of 15°	0.00426
of 10°	0.00190
of 5°	0.00012
of $2\frac{1}{2}$°	0.00003

So that when the circular arc, in which a pendulum vibrates, does not exceed $2\frac{1}{2}$° on each side of the vertical, or 5° in all, the

excess in the length of the time of its oscillations over those in a cycloid does not exceed 9" per day.

274. It has been seen, § 96, that the intensity of gravity varies as we proceed from the surface of the earth, in the inverse ratio of the squares of the distances.

If we call the radius of the earth R, the distance above the mean surface h, the force of the gravity at the surface g, and at the height h, g', we have

$$g : g' :: (R+h)^2 : R^2;$$

whence we obtain

$$g=g'\left(1+\frac{2h}{R}+\frac{h^2}{R^2}\right);$$

or, neglecting the last term,

$$g=g'\left[1+\frac{2h}{R}\right]=g'+\frac{2g'h}{R};$$

and

$$g'=g-\frac{2g'h}{R}. \qquad (292)$$

In the same manner we obtain for the values of the lengths of the pendulums, at the surface, and at the height h,

$$\left.\begin{aligned} l&=l'+\frac{2l'h}{R}, \\ l'&=l-\frac{2lh}{R}. \end{aligned}\right\} \qquad (293)$$

These relations are, however, only true in the case impossible in practice, that the pendulum is raised through the air without resting, as it must, upon some projecting part of the solid crust of the earth. In this event, a local attraction of the mass of ground on which it rests will interfere with the law.

275. The length of the pendulum that vibrates seconds at any place, is proportioned, as has been stated, to the force of gravity at that part of the earth's surface. That is to say, to the difference between the whole force of gravity and the centrifugal force. We may hence obtain the relation between the lengths of pendulums in different latitudes.

If the centrifugal force at the equator be called f, the latitude L, the force of gravity at the pole, or the absolute measure of that force, G, the relative force at the equator g,

$$g=G-f; \quad \text{and } G=g+f;$$

and the centrifugal force at the latitude L, will be, § 100,

$$f \cos.^2 L.$$

Then if g' be the apparent force of gravity at the latitude, L,

$$g'=G-f\cos.^2 L;$$

substituting the value of G from the above equation,

$$g'=g+f-f\cos.^2 L=g+f(1-\cos.^2 L),$$

$$g'=g+f\sin.^2 L. \qquad (294)$$

As the pendulums of the different places have the same ratio as the gravitating forces, we have, if E be the length of the pendulum at the equator, and d the difference between the lengths of the polar and equatorial pendulums, for the difference between the length m of the pendulum at the equator, and at the latitude L,

$$m=E+d\sin.^2 L; \qquad (295)$$

and for the length n, at another latitude L′,

$$n=E+d\sin.^2 L'. \qquad (296)$$

If m and n be given, the lengths at the pole, and the equator may be found; for by subtraction of the foregoing expressions we obtain

$$m-n=(E+d\sin.^2 L)-(E+d\sin.^2 L')=d(\sin.^2 L-\sin.^2 L');$$

whence we obtain for the value of d,

$$d=\frac{m-n}{\sin.(L+L').\sin.(L-L')}; \qquad (297)$$

and d being given, we have the value of E from either of the formulæ, (295) and (296),

$$\left.\begin{array}{l} E=m-d\sin.^2 L \\ E=n-d\sin.^2 L' \end{array}\right\} \qquad (298)$$

The ratio $\frac{d}{E}$ will be the same as $\frac{f}{g}$, which expresses the diminution of gravity from the pole to the equator.

When this ratio is given, the ellipticity of the terrestrial spheroid may be determined; for it has been shown, by writers on physical astronomy, that if this ratio be subtracted from ⁵ of the proportion of the centrifugal force at the equator, to the apparent force of gravity there, the remainder expresses the oblateness or flattening at the poles.

When more than two observations are to be combined, it may be best done by the method of least squares, as may be seen in the Mecanique Celeste, in a paper of Dr. Adrain in the American Philosophical Transactions, and in Sabine's Experiments on the Pendulum.

275. The Simple Pendulum, such as we have assumed it, for the purpose of investigating the theory, cannot exist in practice, for we can neither make use of a body so small as to be considered as a single gravitating point, nor abstract from the weight of the

rod by which it is suspended. Such pendulums as can be actually constructed, are called Compound Pendulums.

In every compound pendulum, there is necessarily a point, in which if all the matter were collected, a simple pendulum will be formed, whose oscillations will be performed in the same time as those of the compound pendulum. This point is called the Centre of Oscillation. Its essential property is, obviously, that the sum of the moments of rotation of the several points of which it is made up, will be equal to the moment of rotation of the whole mass, if situated at the centre of oscillation.

Calling the distance from the centre of oscillation x; the several points of which the body is made up, A, B, C, &c., their respective distances from the centre of suspension, a, b, c, &c., we have for the sum of their moments of rotation, by § 241,

$$(Aa^2+Bb^2+Cc^2+\&c.)\,\frac{u}{a};$$

and for the moment of rotation of the whole mass, if collected in the centre of oscillation,

$$(Aa+Bb+Cc+\&c.)\,x\,\frac{u}{a};$$

whence we obtain

$$x=\frac{Aa^2+Bb^2+Cc^2+\&c.}{Aa+Bb+Cc+\&c.}\,. \tag{299}$$

Hence, the centre of oscillation in a vibrating body is identical with the centre of percussion in a revolving body; and the rule for finding its position may be thus expressed in words: Divide the moment of inertia of the body by its moment of rotation; the quotient is the distance of the centre of oscillation, from the centre of suspension.

If the points, instead of being united by a line devoid of weight, compose a solid body, the same proposition will be true, provided the distances be measured from the several points, perpendicularly to the horizontal line passing through the point of suspension; and if the body, instead of being suspended by a point, is supported on the last mentioned line as an axis, the propositions are still true in respect to it. So, also, if a horizontal line be drawn through the centre of oscillation, every point in it will be at an equal distance from the axis of suspension; and hence any point in the former line, which may be called the axis of oscillation, will have the properties of a centre of oscillation.

From the properties of the centre of gravity, the quantity

$$Aa+Bb+Cc+\&c.$$

is equal to the mass multiplied by the distance of the centre of gravity of the body, from the centre of suspension, or calling the

former, M, the latter, g, to Mg. And if we call the moment of inertia, S, the formula, (299), becomes

$$x=\frac{S}{Mg},$$

and

$$S=Mgx. \tag{300}$$

If the pendulum be suspended by its axis of oscillation, that being parallel to the axis of suspension, the moment of rotation, S′, now becomes, (259),

$$S'=S-m(k'^2-k^2),$$
$$=S-m(k'+k)(k'-k);$$

but $k'+k$ is the distance between the axes of suspension and oscillation, or $=x$, therefore,

$$S'=S-mkx+mk'x;$$

but, by (246),

$$S=mkx;$$

hence,

$$S'=mkx;$$

and the moment of rotation is mk; hence, the distance, x', from the axis of oscillation, when the pendulum is suspended by it, to the line that becomes the axis of oscillation, is

$$x'=\frac{mkx}{mk}=x. \tag{301}$$

277. Thus it appears, that if a compound pendulum be suspended by its axis of oscillation, the axis of suspension becomes the axis of oscillation, or that the centres of oscillation and suspension are convertible points. Therefore, if a pendulum be suspended upon an axis passing through its centre of oscillation, it will vibrate in the same time as when suspended from its ordinary axis of suspension; and conversely, if a pendulum be found to vibrate in equal times when suspended from two different axes, and if one of these be called the axis of suspension, the other becomes the axis of oscillation; and the distance between these axes is the length of a simple pendulum, whose oscillations would be isochronous, with those of the compound pendulum.

278. When the length of a pendulum is spoken of, we do not understand its physical length, or distance between its two extreme ends; but, by this term we understand the distance between its centres of oscillation and suspension, or the length of the identical simple pendulum. The determination of the position of the centre of oscillation of a pendulum, therefore, frequently becomes an important subject of inquiry in practice. This may be effected in the case of all solids formed by the revolution of symmetric curves, by means of the principle in § 276, that the distance be-

tween the centres of suspension and oscillation, may be found by dividing the moment of inertia by the moment of rotation, both in reference to the axis of suspension.

(1). Thus in the case of a straight line, suspended by one of its extremities, the moment of inertia is $\frac{a^3}{3}$, (247), the moment of rotation $\frac{a^2}{2}$; hence, we have for the distance between the centres of oscillation and suspension,

$$l=\frac{2a}{3}. \qquad (302)$$

In the same manner we may obtain, using the formulæ, (253) and (254):

(2). In a cylinder suspended by the middle of one of its bases, a being the length, and, b, the radius of the base,

$$l=\frac{2a}{3}+\frac{b}{2a}. \qquad (303)$$

(3). In a cone suspended by the vertex, the radius of whose base is b,

$$l=\frac{4a}{5}+\frac{b^2}{5a}. \qquad (304)$$

When the cone is a right cone,

$$a=b,$$

and

$$l=a.$$

In which case the centre of oscillation is in the middle of the base.

(4). In a sphere whose radius is r, and which is suspended from a point on its surface, from (256)

$$l=\frac{7r}{5}. \qquad (305)$$

(5). In a sphere whose radius is r, and which is suspended by a line devoid of weight, that unites a point in its surface to the centre of suspension: if c be the length of this line, we have, by using the principles of § 250,

$$l=c+\frac{2r^2}{5c}. \qquad (306)$$

279. It has already been mentioned that pendulums in all the cases in which they can be employed, are resisted by the friction upon the axis of suspension, and by the resistance of the medium in which they move, namely, the atmosphere. The arcs described, therefore, gradually become less and less, until the motion ceases altogether, and the pendulum reaches a state of rest. The resistance at the axis of suspension is usually diminished to the lowest practicable

limit, and hence it becomes unnecessary to apply any correction for it, except so far as its effect becomes apparent in the gradual lessening of the arc of oscillation.

In respect to the resistance of the air, as the velocity of the pendulum is at most but small, that part of it which varies with the square of the velocity may be neglected, and the resistance may be considered as directly proportional to the velocity. Were this absolutely true, although the motion would be retarded, and each oscillation increased in duration, the time of the performance of each would be equally affected, and the isochronism in arcs of a cycloid would not be altered, while in circular arcs the variation in isochronism would be allowed for, in the correction for the extent of the arc. The whole retardation would, if this were true, be allowed for by applying a correction for the buoyancy of the air. The gravity of the pendulum is diminished by its falling through that fluid; and hence, if m represent the absolute density of the pendulum, m', the density of the air, the correction would be

$$\frac{m-m'}{m}. \qquad (307)$$

Such, at least, is the only correction that has hitherto been employed to determine, from the actual oscillations of a pendulum, what would occur were it to move in a vacuum or space void of air. Recent experiments by Bessel, and by Sabine, have shown that this correction is insufficient. Bessel's experiments consisted in making pendulums of similar figures, but of unequal densities, vibrate in air; and in making the same pendulums vibrate in two different media, say air and water. From these, he inferred the difference that would ensue, were the pendulum to move in a space void of air. Sabine, on the other hand, instituted a direct comparison between the oscillations of a pendulum in air, and in such a vacuum as may be obtained by the air-pump. These different modes of experimenting lead to similar conclusions, namely, that the correction necessary to be applied to reduce the motion in air, to that in a vacuum, is in all cases greater than is represented by the formula that has just been stated; and that it depends upon the figure of the pendulum. It must, therefore, be ascertained for every different shape that is given to the pendulum, by actual experiment, for it can hardly be susceptible of reduction to any precise formula. The consequence of this discovery, upon some experiments with the pendulum, will hereafter be cited.

CHAPTER V.

Applications of the Pendulum.

280. The pendulum may be applied to three several important purposes.

(1.) To measure portions of time, or to subdivide the units we derive from astronomic phenomena, into smaller and equal portions.

(2.) To determine the measure of the force of gravity, at different places, and under different circumstances; and thus to enable us to infer the variation in the apparent intensity that is due to the centrifugal force; and the variation in the actual intensity at the surface, that is due to the figure of the earth. Hence the figure of the earth may be inferred.

(3.) The pendulum has been employed as a standard of measure.

281. The pendulum is employed as a measure of time, upon the principle that its oscillations, in very small circular arcs, are isochronous, provided the length of the pendulum do not vary, and the intensity of gravity, at a given place, remains constant. The latter, we have every reason to believe, is unchangeable in the same part of the earth's surface, and it will be seen that there are methods by which the former may be rendered nearly invariable also.

When pendulums are employed as a measure of time, they are adapted to instruments called Clocks. These instruments answer the two-fold purpose:

(1.) Of restoring to the pendulum, at each oscillation, as much force as it loses, in consequence of the resistance of the air, and the friction at the axis of suspension; and

(2.) Of registering the number of oscillations performed by the pendulum. To accomplish the latter object, they are so constructed, that by simple inspection, the interval of time, or number of oscillations that have elapsed since the clock was last observed, can be at once determined. The rate of the oscillations of the pendulum can be changed by altering its length; and thus by comparing the clock with astronomic observations, the day may be divided into the usual number of conventional parts. The greater number of clocks have pendulums that oscillate once in each second of time; and when we speak of the length of a pendulum at a given place, we mean one whose beats are an ex-

act second, or $\frac{1}{86400}$ part of a mean solar day. Some clocks have pendulums that beat half seconds, and others again oscillate but once in two seconds. According to the principles in § 271, the pendulums of the former must have no more than one fourth of the length, and of the latter, must be four times as long as the second's pendulum.

It was at one time attempted to make the pendulums, used as measures of time, vibrate in arcs of a cycloid. This has been abandoned, in consequence of the mechanical difficulties of making the cycloidal cheeks, to which it should adapt itself: of the impossibility of obtaining a flexible string of invariable length; and of the doubt that must exist in pendulums of any convenient form, in respect to the position of the point, the centre of oscillation, that ought to describe the curve.

282. The pendulum of a clock is usually composed of a weight, or bulb, of a lenticular form, adapted to an axis of suspension by a metallic rod. As both the rod and the bulb are liable to vary in magnitude, by variations in temperature, the position of the centre of oscillation, and consequently the length of the pendulum, must be undergoing perpetual alterations. Thus the law of isochronism will be no longer true, and the clock will not move at an uniform rate, nor mark equal divisions of time. To obviate this defect, various modifications of the original simple form have been contrived, which are called Compensation Pendulums. Their general principle is identical, and consists in making the pendulum of two substances that expand in opposite directions, in such a manner as to keep the centre of oscillation at a constant distance from the axis of suspension.

Various methods have been proposed and employed for this purpose:

(1.) The rod of a clock pendulum is often attached to a short piece of flexible metal or spring; this may be griped by two plane surfaces, pressed against it by screws; and its effectual length will be measured from the lower edge of these surfaces, provided the pressure be sufficient to support its whole weight. Now if a bar of the same metal with that which forms the rod of the pendulum, and of an equal length, be adapted to the back of the clock-case; and if it be so applied to a horizontal arm, attached to the spring that bears the pendulum, that it shall in its contractions and expansions cause the spring to slide between the surfaces on which it rests; every change in the length of the pendulum rod, under the influence of temperature, would be exactly counteracted. The objection that applies to this method is, that if the friction of the surfaces against the spring be sufficient to bear the whole weight of the pendulum, it will interfere with the ac-

tion of the compensation; while if it be not, the effective length of the pendulum is no longer determined by them. Such is the plan of compensation proposed by Dr. Fordyce, which has the advantage of great simplicity, although in consequence of the defects that have been stated, it is not perfect.

(2.) The rod and bulb of a pendulum being separate bodies, and the former generally passing through the latter, which only rests upon a projecting part, it occurred to Graham, that were they to be made of two different substances, the expansion of the rod downwards, might be compensated by the expansion of the bulb upwards. There is not, however, a sufficient difference in the expansibilities of the solid metals, to allow this principle to be carried into effect by means of them. But mercury expands about 16 times as much as steel; and hence, could the rod be made of the latter metal, and the bulb of the former, in a ratio of dimension of about 16 : 1, a compensation might be effected. To form the bulb, the mercury is placed in a cylindrical jar of glass; and in order to support the jar from beneath, the rod is divided into two branches, connected at the lower end by a plate, having thus a shape analogous to a stirrup. This pendulum, called the Mercurial Pendulum of Graham, is perhaps the most perfect of all compensations. Its original adjustments are, however, more difficult than those of some others we shall describe; and if it should be removed from the place where it was originally adjusted, the experiments must be again repeated. It is, therefore, only used in fixed observatories.

To investigate the relation between the length of the column of mercury in the vase of the mercurial pendulum, and the whole length of the rod:

Let l be the whole length of the steel rod; s the lineal expansion of steel; y the unknown length of the column of mercury; m the cubic expansion of mercury; g the lineal expansion of glass. Suppose that the conditions of the problem require the centre of gravity of the mercury to remain in a constant position, which would be true, provided the position of the centre of gravity of the rod also remained constant. The distance of the centre of gravity of the mercury in the vase from the bottom, or extremity of the pendulum, is $\frac{y}{2}$, and the distance, L, of its centre of gravity from the axis of suspension, is,

$$L=l-\frac{y}{2};$$

and for a perfect compensation, this expression must be a constant quantity, however l and g vary under the influence of temperature.

If r be the radius of the vase that contains the mercury at the standard temperature; V, the volume of mercury, will be

$$V=\pi r^2 y. \tag{308}$$

At t degrees, the radius of the vase will become

$$r(1+gt);$$

y will also vary and become y'; and, at the same time, the volume of mercury will become

$$v(1+mt);$$

hence

$$V(1+mt)=\pi r^2(1+gt)^2 Y'; \tag{309}$$

dividing this by the equation (308), we obtain

$$1+mt=\frac{(1+gt)^2 y'}{y}; \tag{310}$$

whence we have

$$y'=y\frac{(1+mt)}{(1+gt)^2}, \tag{311}$$

which is the value of the height of the column of mercury at the new temperature, t. But as g is but a small quantity, its second power may be neglected without any sensible error; and the expression will become

$$y'=y+y(m-2g)t. \tag{312}$$

The constant distance, L, will at the same temperature be

$$L=l-\frac{y}{2}+lst-\frac{y'}{2}; \tag{313}$$

and substituting the value of y',

$$L=l-\frac{y}{2}+\left[ls-\frac{y}{2}(m-2g)\right]t; \tag{314}$$

and as this must be equal to the first value, the variable term affected by t must be $=0$, or

$$ls-\frac{y}{2}(m-2g)=0; \tag{315}$$

whence we have for the value of y,

$$y=l\frac{2s}{m-2g}; \tag{316}$$

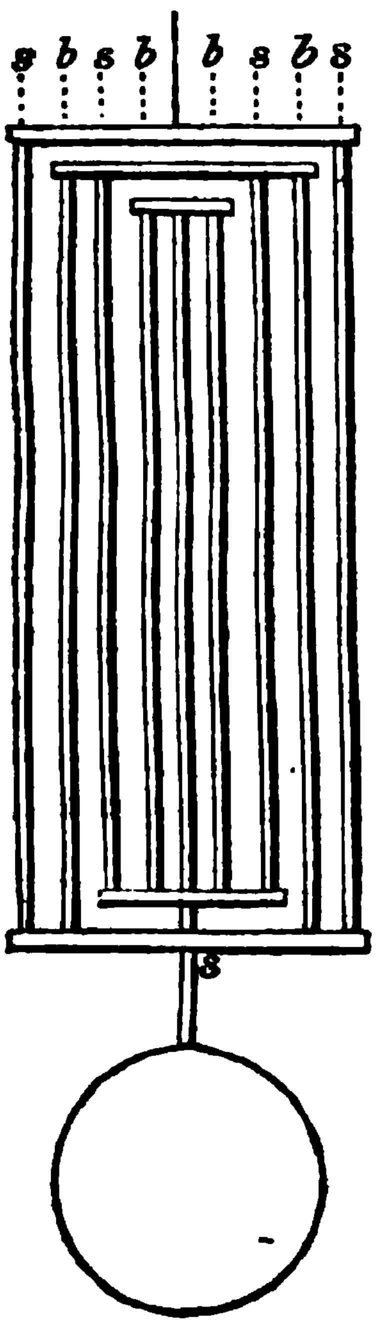
taking the exact expansions of the three substances between the boiling and freezing points, we have

$$s=0.00124,$$
$$m=0.01848,$$
$$g=0.00088;$$

and substituting these values, we obtain

$$y=\frac{l}{6.75}.$$

(3.) Harison, finding that Graham had failed in applying the solid metals as a compensation to the two separate parts of the pendulum, abandoned that method, and sought for an application of them to the rod alone. Inferring from experiments on the expansion of brass and steel, that their relative expansion was nearly in the ratio of 3 : 5, he planned a frame of which the following figure will give an idea. It was composed of nine parallel bars, and from a fancied resemblance, was called the Gridiron Pendulum. The five bars marked *s*, are of steel, the four marked *b*, are of brass; the centre rod of steel is fixed at top to the cross-bar, connecting the two contiguous brass rods, but slides freely through the two lower bars that cross its direction. This centre rod bears the bulb. The remaining rods are fastened to the cross pieces at both ends, and the outer and upper cross-piece is attached to the axis of suspension. It will be apparent, from inspection, that the steel rods will, in their expansion, tend to lengthen the pendulum, while the brass rods will, in theirs, tend to shorten it. If these two expansions exactly counterbalance each other, the length of the pendulum will remain invariable. The four brass rods act by pairs, and, therefore, as if there were but two; and no more than three of the five steel rods are to be considered, for four of them also are connected in pairs. Hence, if the expansion of the sum of three of the steel rods added to the expansion of the rod that connects the gridiron frame to the axis of suspension be equal to the expansion of two of the brass rods, the condition of compensation is fulfilled.



To investigate the lengths that should be given to the brass rods in the gridiron pendulum of Harison:

Let L be the distance from the axis of suspension to the extreme end of the pendulum; and let the condition of compensation be that this length shall remain constant. Let l be the common length of the outer pair of steel rods; l' that of the inner pair; a the distance from the gridiron frame to the axis of suspension

occupied by a steel rod; and b the length of the steel rod that bears the bulb, occupying the middle of the frame. Let λ and λ' be the several lengths of the two pairs of brass rods. We have from the construction of the apparatus,

$$L = a + l + l' + b - \lambda - \lambda'; \qquad (317)$$

if we call the expansions of brass and steel, B and S, we have for the constant value of L, after a change of temperature of t degrees,

$$L = a + l + l' + b - \lambda - \lambda' + [(a + l + l' + b) \cdot S - (\lambda + \lambda') \cdot B]; \qquad (318)$$

and in this equation, the variable term affected by S and B $= 0$, or

$$(a + l + l' + b)\, S - (\lambda + \lambda')\, B = 0;$$

but we have from our first equation,

$$a + l + l' + b = L + \lambda + \lambda';$$

and substituting this value of the foregoing, we obtain

$$(L + \lambda + \lambda)\, S - (\lambda + \lambda')\, B = 0; \qquad (319)$$

whence

$$\left.\begin{aligned} (\lambda + \lambda') \cdot (B - S) &= L\,S \\ \text{and} \qquad \lambda + \lambda' &= \frac{L\,S}{B - S} \end{aligned}\right\} \qquad (320)$$

If we take the approximate ratio of expansion used by Harison, we have

$$\lambda + \lambda' = \frac{3\,L}{2},$$

or for the joint length of the two pairs of brass bars, one and a half times the whole distance, from the axis of suspension to the extremity of the pendulum.

These investigations may serve as a guide in the construction of these two species of pendulum, but they are obviously inexact. Nor is it necessary that they should be more accurate; for the adjustment of each must finally be made by experiment; and reference must be had, not only to the pendulum itself, but to the clock; for the action of the clock on the pendulum will be affected by changes of temperature, and the pendulum must meet this, as well as its own variations.

In the gridiron pendulum, as has been seen, five rods, three of steel and two of brass, are sufficient for the purpose of compensation; the other four are added for the purpose of making it symmetric, and causing the line of direction of its centre of gravity to pass, when the pendulum is at rest, through the centre of magnitude of the bulb.

(4.) The expansion of an alloy of 8 pts of zinc and one of tin, is to that of steel nearly as 2 : 1. Hence, a smaller number of bars of these two substances will furnish a compensation. Thus

the pendulum applied by Breguet to his clocks, is composed of no more than five rods, three of steel and two of zinc; two of the steel rods and the two of zinc being combined in pairs, so that it may be considered as composed, so far as the principle of compensation is concerned, of no more than three rods.

The pendulum of Harison has been improved by Troughton, who has substituted for the two pairs of brass rods, two cylinders of the same metal sliding one within the other, to which the iron rods are attached. The principle is obviously the same, but it has some advantages over the original form, inasmuch as it is less liable to external injury, and the brass tubes will not bend under the upward pressure of their expansion, to which rods of the same metal are liable.

Such are the more important of the forms of compensation pendulums. The number and variety of those that have been proposed is very great, and it would occupy too much space to enter into a description of them. Those who feel a curiosity to examine their different structures, will find an admirable paper on the subject by Kater, in the work on Mechanics, which bears the joint names of himself and Dr. Lardner.

283. In order to determine the intensity of gravity, by means of the pendulum, it becomes necessary to measure its length; that is to say, to determine the distance between its axis of suspension and centre of oscillation Two principal methods are now employed for this purpose, those of Borda and Kater.

284. In Borda's method, the experimental pendulum, from the measure of which the length of the second's pendulum is to be inferred, is composed of a sphere of platinum, suspended by a slender wire of iron, from a knife-edge of steel resting on plane surfaces of polished agate. This form, employing the densest of known substances, and the slenderest wire that is sufficient to bear it with safety, approaches as nearly as possible to the hypothetical simple pendulum.

Its length, considered as a pendulum, or the distance between its centres of suspension and oscillation, is determined by calculation from its total physical length, obtained by actual measurement. To effect this measurement, the pendulum is rendered stable, by screwing up from beneath, a cup-shaped vessel, that just catches the ball of the pendulum, without bearing any part of its weight. A scale of iron is then applied to it, on which the physical length is marked. An improved method consists in screwing up from beneath a plane of polished steel, until it just touches the platinum sphere; the pendulum is then removed, and to its place is adapted a scale, by means of knife-edges simi-

lar to those of the pendulum. This scale is composed of two parts, one of which is firmly fastened to the knife-edge, and is shorter than the pendulum; the other slides upon this, and is moved by a screw. The scale being thus placed, the moveable part is depressed by means of the screw, until it just touches the steel plate; the length of the two portions united, that is to say, of the part fixed to the knife-edge, added to that of the projection of the moveable part, is of course just equal to the physical length of the experimental pendulum.

The theoretic length, or the distance between the axes of suspension and oscillation, is next deduced, upon the principle of its being a sphere suspended by a line void of weight, by the formula (306),

$$l=c+\frac{2r^2}{5c}.$$

Such at least would be the principle, were the wire and sphere the only parts of the pendulum, and the former devoid of weight. As, however, neither of these is true, particularly as parts must be adapted to attach the wire to the knife-edge and to the sphere, a much more complex formula must be used in practice. We refer for this to the "*Base du Systeme Métrique,*" and to Delambre's Treatise on Astronomy.

In the original apparatus of Borda, the length of the experimental pendulum was four times the length of the second's pendulum. The time of its oscillation was determined by a method called that of Coincidences. For this purpose, the pendulum was suspended upon knife edges, resting on planes of agate, in front of a well-regulated astronomical clock, having a compensation pendulum. The knife edge was moved upon the agate planes, until the wire of the experimental pendulum, as viewed through a small telescope, placed at the distance of 12 or 15 feet in front of the clock, exactly coincided with the centre of the circle, bounding the lenticular bulb of this clock pendulum. This point was marked by drawing two black lines on a white ground, making each an angle of 45° with the horizon; a black screen was so placed as to hide just half the wire of the experimental pendulum. The two pendulums being set in motion, an observer placed at the telescope, would see the wire, and the point marked upon the clock pendulum disappear behind the edge of the screen, at each of their alternate oscillations. If, when first observed, they did not pass the edge of the screen at the same instant of time, they would, provided the one were not exactly four times as long as the other, gradually approach, until both would disappear at the same instant. The time marked by the clock is then noted, as the instant of the first coincidence.

It is usual to make the experimental pendulum a little longer than four times that of the clock; hence the former makes a little less than one oscillation for every two of the latter. After the interval of four seconds, the wire and the cross will be again in the field of the telescope at the same time, but the cross will precede the wire. At each successive interval of four seconds, the distance at which they pass each other, will increase until the interval of the times of their respective disappearances amounts to 1″. After this they will approach, until they again pass the eye, and disappear behind the screen at the same instant, which is noted as that of a second coincidence. During this interval, the clock pendulum will have gained two oscillations upon the experimental pendulum; that is to say, the number of the oscillations of the experimental pendulum, will have been one less than half the number of seconds marked by the clock; the latter number is obtained by simple inspection of the dial of the clock.

The observation of the coincidences is continued, until the experimental pendulum has lost too much of its motion to render them easily distinguishable, and the record of the times is collected in a set.

285. In order to reduce the oscillations performed by the experimental pendulum, to a cycloid, or an infinitely small circular arc; the extent of the arcs of vibration on each side of the vertical, are observed at each coincidence. The correction is applied upon the principle of the formula, (285), in which the last terms of the series of (83), are neglected. This, in very small arcs, becomes very nearly

$$\frac{\sin.^2 a}{16}.$$

It is more convenient, however, to apply the correction to the whole set, in which case the mean of the first and last arcs of vibration may be taken. But as the arcs decrease in fact in a geometric progression, a more correct formula has been calculated by Borda, which is as follows:

$$m = \frac{\sin. (a+a') \sin. (a-a')}{32\,\mathrm{M}\,(\log. \sin. a - \log. \sin. a'}; \qquad (321)$$

in which a and a' are the greatest and least amplitudes of the oscillations; and M, the modulus of the tables of logarithms, $=2.30258509$.

The pendulum vibrating in air, the number of oscillations it is observed to make, must be corrected for the buoyancy of the air, which is done by the formula (307). No correction has hitherto been applied for the variation in the arc's resistance, growing

out of the figure of the pendulum, detected by Bessel and Sabine, as stated in § 279.

As the temperature of the apparatus may vary during the series of coincidences, the length obtained by measurement will probably not be the same as that at which any one of the coincidences has occurred, and the latter will differ from each other. The wire being capable of expansion and contraction, by changes of temperature, it will be necessary to reduce the whole to some standard temperature. So also must the length of the rod, by which it is measured, be corrected for the temperature at which that part of the operation is performed.

The corrections for the expansion by temperature, may be obtained from the tables of the expansion of the metals that are to be found in authors on physical subjects: or, they may be obtained from experiments on the varying rate of the pendulum's own oscillations, at different degrees of heat. The last method was used by Sabine, and is the best, inasmuch as the actual expansion of the apparatus is obtained, instead of the mean expansion of the species of substance employed.

When the length of the experimental pendulum is known, and the number of oscillations it performs in a given time, say in a mean solar day, is determined and reduced to a cycloidal arc and a vacuum; the length of the pendulum that would vibrate seconds, may be at once obtained; for it has been shown, § 272, that the respective lengths of pendulums, are inversely as the squares of their numbers of vibration in equal times; hence, as there are 86400 seconds in a mean solar day, if L be the length of the experimental pendulum, corrected for temperature; N the number of vibrations it performs in a mean solar day, corrected for the arc of vibration, and for the buoyancy of the air: the length l of the pendulum that vibrates seconds, will be

$$l = L\frac{N^2}{(86400)^2}.$$

It still remains that the length thus obtained should be reduced to the mean surface of the earth, or to the level of the sea. The altitude of the place of observation above this surface, must, therefore, be determined, and the reduction performed by the formula, (293),

$$l = l' + \frac{2l'h}{R}.$$

The form and nature of the ground will influence this result; and hence, when the elevation is considerable, the length thus obtained will not be that which would be found at the level of the sea. It has been stated by Dr. Young, in the Transactions of the Royal

Society of London, that if the station be upon a table land of a density equal to ⅔ds of the mean density of the earth, the diminution of the force of gravity will be no more than one-half of what is due to the height above the level of the sea; and that in the most unequal country, there will be at least ¼th to be deducted from the correction obtained by the above formula, (283). In Sabine's investigation, this correction has been multiplied by the constant coefficient, 0.6.

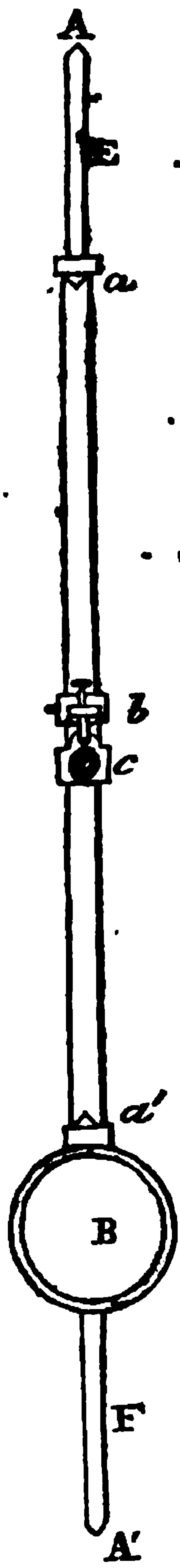
It has also been discovered by Sabine, whose remark has been confirmed by Biot, that the nature of the ground on which the experiment is performed, affects the length of the pendulum in all places. The attraction of gravitation being greater upon dense earthy substances than it is upon rare.

As an instance of the effect of elevated ground, we shall cite the observations of Carlini, at Mount Cenis. The pendulum, measured at the top of the mountain, and reduced to the level of the sea by the usual method, had a length in French measure of $0.^{m}993708$; while a pendulum, for the same latitude deduced from the pendulum of Bourdeaux, would not have been longer than $0.^{m}993498$. The difference of $0.^{m}000210$ is, therefore, due to the local attraction. This observation may be cited as being among those whence the density of the earth has been inferred.

The method of Borda has been improved by Biot, and the apparatus rendered more convenient. The length of the experimental pendulum has been reduced to one that differs but little from that of the pendulum of the clock: a copper wire has been substituted for one of steel, as less liable to rust; and the whole apparatus enclosed in a glass case, to render it less exposed to the action of currents of air, and to sudden changes of temperature. With this diminished length, the pendulum that moves fastest still gains two oscillations upon the other, between each coincidence.

286. The method of Kater is founded upon the principle, §276, that the centres of suspension and oscillation are convertible points; and conversely, that if a pendulum vibrate in equal times, upon two different parallel axes, one of these has the relation to the other of the axis of suspension to that of oscillation. If then a pendulum be taken, into which two knife edges, turned in opposite directions, are inserted; if the distance between these knife edges is very nearly that which can be estimated to exist between the centres of oscillation and suspension; and if a moveable weight be adapted to the rod, this weight may be so adjusted by trials, that the pendulum shall oscillate in equal times, when suspended by either axis.

The form of the pendulum of Kater is such as is represented beneath.



The rod AEFA', is a bar of hammered brass, to which is adapted the bulb B, of cast brass, and of a form no more different from a cylinder than is necessary for the convenience of casting. The knife edges made of wootz, are represented at *a* and *a'*; they are formed of two planes, meeting at an angle of about 60°. The moveable weight is in two parts, and slides on the bar AA'. The part *c*, is capable of being firmly fixed to the bar by a screw, whose head is represented; the part *d*, is attached to the part *c*, by a screw of a fine thread, by means of which a slow motion may be given to *d* after *c* is made fast. The pendulum is brought nearly to its adjustment, by sliding the whole weight along the rod; *c* is then firmly fastened by its screw, and the adjustment is completed by the slow motion of *d*.

The portions E and F, of the rod A A', were in the original apparatus of Kater, made of blackened wood.

The observations of the coincidences, by means of which the pendulum is adjusted, and whence the value of its oscillations is determined for calculation, when adjusted, are made as follows: The clock pendulum being at rest, a small telescope is placed directly in front of it, and the experimental pendulum is suspended in such a manner that one of its blackened terminations, just hides a white circular spot, drawn upon the lens of the clock pendulum, and concentric with it. The diaphragm of the telescope has two plates with vertical edges; these are pressed forward by screws, until they appear in optical contact with the blackened bar. When the pendulums are set in motion, during the greater part of the oscillations, the bar and the white spot will both be seen; but, from time to time, for a few contiguous oscillations, the bar will wholly hide the white spot. Kater took, as the instant of coincidence, the beat of the pendulum that followed the first passage in which no part of the white spot was seen. This method is objectionable, inasmuch as the same observer will, under different circumstances of light, continue to

see the spot a longer or shorter time before it is wholly obscured. The tact of observation, even in the same observer, will differ at different times, and will improve by practice; while, when the experiments of different observers are compared, a marked difference will be found to exist in their powers of vision.

For these reasons, Sabine, in his experiments, adopted the plan of observing, not only the disappearance, but also the reappearance of the white disk; noting the second succeeding the former, and preceding the latter: the mean of the two was taken as the instant of coincidence. The author had a good opportunity of testing the value of these two different methods, when associated with Sabine, in the experiments to determine the length of the pendulum at New-York. Had the experiments made by the two observers been compared upon the plan of Kater, a considerable discrepancy would have ensued; but when compared by the method of Sabine, the results were nearly identical.

The length of the experimental pendulum, in Kater's method, is determined directly, without reference to calculation, by measuring the distance between the knife edges. This is effected by means of a scale furnished with powerful microscopes; to one of these a micrometer is adapted. With this apparatus, the 10,000th part of an inch becomes a measurable quantity. The method of Kater requires the same corrections and reductions as that of Borda; thus the time of oscillation must be corrected for the arc of vibration, and for the buoyancy of the air; the length must be corrected for the temperature, and the second's pendulum, calculated from the observations, reduced to the level of the sea.

The method of Kater is liable to one objection, namely, that in conformity with the views of Bessel, it will be unequally resisted by the air, when suspended by its different knife-edges. A modification of the apparatus proposed by Bailey, is less liable to this objection: this pendulum is a simple bar, without a bulb, and the adjustment is effected by filing away those portions whose weight is in excess.

287. In order to determine the measure of the force of gravity at any given place, when the length of the second's pendulum is known, we have the formula (289)

$$g=l\pi^2.$$

At New-York, the length of the pendulum, or l, is 39.1 in. nearly, or 3 ft. 2583, hence

$$g=32 \text{ ft. } 1576,$$

or nearly $32\frac{1}{6}$ feet.

A heavy body will, in consequence, fall in vacuo during the first second of time, § 271, through a space equal to $16\frac{1}{12}$ feet.

288. It has been stated that the apparent intensity of gravity, or the difference between its absolute force, and the diminution growing out of the earth's rotation, may be immediately deduced from a measure of the second's pendulum. In the method of Borda, the experimental pendulum is made to vibrate in the several places in which it is desired to ascertain this quantity ; but as the length of the suspending wire may vary, it becomes necessary to go through the whole process at each station.

In the method of Kater, as the distance between the knife edges is invariable, except by changes of temperature, the influence of which is known, one careful measurement will suffice for any number of stations. The original pendulum may, therefore, be carried from station to station, and its coincidences observed. A direct comparison between those observed at different places, gives an immediate determination of the length of the pendulum that would oscillate in a second, at the several different stations.

The method has been still farther improved by its author. A pendulum, having but one knife edge at the usual point of suspension, is suspended in front of the clock, in the place where the original experiment was made, with the pendulum with convertible axes. The rate of its oscillations being determined, its length can be calculated by the formula (290), from that of the original pendulum. It may then be carried to other places, and the length of the pendulum of the place determined from the rate of its oscillations, in the same manner. In this way, Kater himself determined the length of the second's pendulum at the more important stations of the British Trigonometrical Survey. Sabine has since employed the same method, at a variety of stations, from Ascension, in lat. 7°, 55', 48", S. ; to Spitzbergen, in lat. 79°. 49'. 58". N.

This part of Kater's method, as applicable to observations, at different places, is much more convenient than that of Borda, even as improved by Biot. It may also be performed in a much less time ; thus, for instance, Biot was engaged for three months at Unst, in completing his measure of the pendulum, while Kater effected his in three weeks.

289. The relation between the lengths of the pendulum, at different places, may also be determined by means of a clock furnished with a pendulum whose rod is not liable to have its dimensions changed by transportation, except in consequence of variations of temperature. Such a clock was used in the first

expeditions of Parry and Ross, and the absolute length was ascertained, by comparison with the original experiment of Kater.

The method of Kater is still imperfect, inasmuch as the length determined at the original station, and therefore at all others, still rests upon his own single experiment; and it has not yet been ascertained, how far it is possible for the same experimenter, or another equally well qualified, to reproduce an identical result. Until this question be settled, it must remain questionable whether the differences in the measure of the pendulum at the same place, by the two different methods of Borda and Kater, arise from the methods themselves, or are involved in the original determination on which the results of the latter method are founded.

290. By the use of these two methods, the pendulum has been measured in various places in both hemispheres, by Kater, Sabine, Biot, Bouvard, Matthieu, Arago, Chaix, Freycinet, and Duperrey. Some of these results are to be found in the following

TABLE.

STATION.	NORTH LATITUDE.	Height above the Sea.	PENDULUM. At the Station.	PENDULUM. Reduced to the level of the Sea.
St. Thomas, . . .	0°.24′.41″	21ft	39.02069	39.02074
Maranham, . . .	2 .31 .43	77	39.01197	39.01214
Sierra Leone, . .	8 .29 .28	190	39.01954	39.01997
Trinidad, ...	10 .38 .56	21	39.01879	39.01884
Jamaica, ...	17 .56 .07	9	39.03508	39.03510
Formontera, . . .	38 .39 .56	606	39.09176	39.09325
New-York, . . .	40 .42 .43	67	39.10153	39.10168
Bourdeaux, . . .	44 .50 .26	56	39.11282	39.11295
Paris, ...	48 .50 .14	230	39.12843	39.12894
London, ...	51 .31 .08	92½	39.13908	39.13929
Leith, ...	55 .58 .39	69	39.15540	39.15546
Unst, ...	60 . 5 .26	30	39.17145	39.17151
Hammerfest, . . .	70 04 .05	29	39.19512	39.19519
Spitzbergen, . . .	79 49 .58	21	39.21464	39.21469

A part of the observations that have been made in the southern hemisphere will be found in the following

TABLE.

STATION.	SOUTH LATITUDE.	PENDULUM REDUCED TO THE LEVEL OF THE SEA.
Ascension, . . .	7°.55'.48"	39.02410
Bahia, ...	10 .38 .56	39.02435
Isle of France, . .	20 .09 .19	39.04793
Port Jackson, . .	33 .51 .39	39.08049
Malouine Islands .	51 .31 .44	39.13695

The general inference of Sabine, from a combination of his own experiments with those of Kater, and those made at the stations of the French trigonometric survey, is that the length of the equatorial pendulum =39 in. 01569; the increase of its length from the equator to the pole =0 in. 20227. The formula, (296), for the length l', at any intermediate latitude L, becomes (296),

$$l'=39.01569+0.20227 \sin.^2 L.$$

A more recent French deduction, into which the observations of Duperrey and Freycinet enter, gives

$$l'=39.01741+.23505 \sin.^2 L.$$

The result obtained by Sabine gives, for the oblateness of the terrestrial spheroid, $\frac{1}{288}$; and for the centrifugal force at the equator, $\frac{1}{294}$th part of the whole force of gravity. The last deduction gives for the centrifugal force, $\frac{1}{287}$. The centrifugal force is usually stated at $\frac{1}{289}$, which lies between the above inferences; and this is the value that we shall employ.

291. It had, until the discovery of the influence of local attraction by Sabine, been generally concluded, that the pendulum vibrating seconds in a given latitude, at the level of the sea, was a constant and invariable quantity. Its length is also capable of comparatively easy determination; as all the observations connected with its measure may be made within the space of a few weeks. Hence it has been proposed as a standard of measure. Were the first inference true, and were the reduction to the level of the sea independent of local influence, no better method could well be devised for the purpose of re-establishing standards of lineal measure that have been lost, or are suspected of being altered by age. It has also been proposed to use the length of the pendulum not only as the standard, but as the unit of lineal measure. This, however, is objectionable, except in the case where

the customary unit of a country differs but little from the length of the pendulum. Such is the case with the measure of Denmark; and hence, under the auspices of Schumacher, a system of weights and measures has been formed in that country, of which the pendulum forms both the unit and the standard.

In cases where the difference between the unit in actual use and the length of the pendulum is considerable, it is better to retain the ancient unit, and define its relation to the pendulum. For this purpose, it will be evident from what has been said in relation to the influence of local circumstances, that it will not be safe to use the pendulum of a given latitude; but that the only admissible method is to take the pendulum measured in a particular place as the standard.

When a unit of lineal measure has been defined, in relation to some standard existing in nature, its square will serve as a unit of measures of surface; its cube, or some aliquot part, as the unit of measures of capacity; and the weight of its cube, filled with pure water at some given temperature, will furnish a unit of measures of weight. It has, however, been found more easy to determine the weight of water that a measure of capacity will hold, than to ascertain its cubic contents; and hence, in some systems, the unit of capacity has been defined by declaring what number of the units of weight it shall contain.

292. This being premised, we shall proceed to describe the principles upon which a reform has been effected in the standards of England, and of the State of New-York, in both of which the pendulum has been assumed as the basis.

The standard of measure in Great Britain is the pendulum, vibrating seconds in a cycloidal arc, in a vacuum, and at the level of the sea, in the latitude of London, 51° 31' 08" N.

The unit of measures of length is the Yard of such magnitude that the pendulum shall measure, 39 in. 13929. The yard is divided into three feet; the foot into twelve inches; and for cloth measure the binary subdivision is permitted. Greater measures of length are multiples of the yard, derived as in the ancient system.

The square of the yard, or of any other unit of length, may be used as a unit of superficial measure.

The standard temperature, to which measures of length or surface are to be reduced, is 62° of Fahrenheit's thermometer, and the material of which the standard yard is made is brass.

The unit of weight is the Troy Pound, of such magnitude that a cubic inch of water, at 62°, weighs 252 grs. 458, there being 5760 grs. in this pound.

The avoirdupois pound is also used, and is defined as being equal to 7000 grs. Troy.

The unit of measures of capacity is the Gallon, which is a vessel that holds exactly ten avoirdupois pounds of water, at the temperature of 62°.

The bushel holds eighty pounds of water at the same temperature.

293. The standard of the state of New-York is the pendulum vibrating seconds in a cycloidal arc, and in a vacuum in Columbia College in the city of New-York.

The unit of lineal measure is the Yard, which is of such magnitude as to bear to the pendulum the proportion of 1,000000 to 1,086158.

Its usual subdivisions are allowed to be employed; and its standard temperature is that of melting ice. It is identical with the present British standard yard, which has been restored in its magnitude to that used previous to the revolution, and which had continued in use in the State of New-York; but in the comparison, each is to be taken at its own standard temperature. The standard temperature of the English system is 62° Fahrenheit; of that of the State of New-York, 32°.

The unit of measures of weight, is the Avoirdupois Pound, of such magnitude that a cubic foot of pure water, at its maximum density, shall weigh 1000 oz. or 62½lbs.

The unit of dry measures of capacity, is the Gallon, a vessel of such magnitude as to hold exactly 10lbs. of pure water, at its maximum density. The bushel, therefore, holds 80lbs.

The unit of liquid measure is also a gallon, containing eight pounds of distilled water, at its maximum of density. The adoption of this unit, was a deviation from the original plan, which contemplated but one set of measures of capacity for solids and fluids, and it has impaired the symmetry of the system.

294. To the English system, it is to be objected: that it assumes for its standard the pendulum of a particular latitude, which will not be constant, in consequence of local circumstances; that the determination on which the length of this standard has been performed in a private building, the house of Mr. Brown; that it retains two units of weight, of the same denomination, but of different magnitudes; and that its standard temperature is wholly arbitrary, founded on no natural phenomenon, and dependent upon a conventional thermometric scale. The mode of defining its unit of weight, moreover, involves a fractional quantity, and the bulk of water employed in the determination, namely, a cubic inch, is too small.

To the system of the State of New-York none of these objections apply, except so far as relates to the double system of measures of capacity. The standard is the pendulum of a particular place; and that, so far as is known, is invariable; that place is a public building, readily accessible; the standard temperatures are defined by physical states of water, in respect to which there can be no error, and which are independent of thermometric scales. The unit of weight is determinable from a bulk of water of sufficient magnitude.

295. The French system of weights and measures has for its standard a quadrant of the meridian. The unit of measures of length is the Metre, which is a ten millionth part of the quadrant.

The unit of superficial measure, is the Are, a surface 10 metres each way, or 100 square metres.

The unit of measures of capacity, is the Litre, a vessel containing the cube of a tenth part of the metre.

The unit of weight is the Gramme which is equal to the weight of the cube of the hundredth part of the metre, filled with distilled water, at its maximum of density, or to the 1000th part of the weight of a litre of water.

The standard temperature of the measures of length is that of melting ice.

The whole of the divisions and multiples of the units were decimal, and the principal of nomenclature adopted, was to prefix the Greek numerals to the decimal multiples, and the Roman numerals to the decimal subdivisions of the units.

Thus the measures of length are,

Myriametre	=	10000 metres.
Kilometre	=	1000 metres.
Hectometre	=	100 metres.
Decametre	=	10 metres.
Metre	=	1 metre.
Decimetre	=	$\frac{1}{10}$ metre.
Centimetre	=	$\frac{1}{100}$ metre.
Millimetre	=	$\frac{1}{1000}$ metre.

The measures of Surface are

Hectare	=	10.000 sq. metres.
Are	=	100 sq. metres.
Centiare	=	1 sq. metre.

The measures of Capacity are

Kilolitre	=	1000 litres.
Hectolitre	=	100 litres.
Decalitre	=	10 litres.
Litre	=	1 litre.

Decilitre	=	$\frac{1}{10}$ litre.
Centilitre	=	$\frac{1}{100}$ litre.

The weights are

Myriogramme	=	10000 grammes.
Kilogramme	=	1000 grammes.
Hectogramme	=	100 grammes.
Decagramme	=	10 grammes.
Gramme	=	1 gramme.
Decigramme	=	$\frac{1}{10}$ gramme.
Centigramme	=	$\frac{1}{100}$ gramme.
Milligramme	=	$\frac{1}{1000}$ gramme.

296. No system can be imagined more perfect and beautiful, in a scientific point of view, than this system of the French nation. It is founded upon a standard existing in nature, and invariable, and which is susceptible of determination to such a degree of exactitude, that no probable error that can, in the present state of science, be committed in the measure of degrees, will affect the small fraction of the standard that forms the unit of length. From its decimal division, it is exactly consistent with our usual system of arithmetic; and its nomenclature is systematic, and of easy recollection. Still it is not without fault, even in a scientific point of view. The measures of length are incapable, for instance, of application to astronomic purposes, in which we use the semidiameter of the earth, and not its quadrant as the unit; and these two magnitudes are incommensurable. Neither are we aware that a measure of the meridian in other countries, particularly in our own hemisphere, would reproduce the same magnitude for the quadrant that was obtained in France. The measure of a sufficient arc, whence to determine the length of a quadrant, is an operation of great expense, and would occupy a long time. Hence, in presenting the types of the units to the National Assembly, the commission propose to verify them, if suspected of alteration, and reproduce them, if lost, by reference to the pendulum of the Observatory of Paris; thus recurring to the same natural standard that had been rejected by them in the outset. The metre is, therefore, after all the labour that was expended in its determination, no more than a conventional length, whose relation to the second's pendulum of a particular place is well determined. It has also been found impracticable to introduce the decimal division into the measure of angles; and after strenuous attempts for that purpose, and the laborious construction of new tables, even the astronomers of France have returned to the ancient division of the circle.

The objections, in a practical point of view, are more numerous, and have been found insuperable. Thus, however well-cal-

culated for scientific purposes, and even for those of commerce, the decimal multiples of the unit may be, decimal subdivisions have been found unsuited for the purpose of retail traffic; for this object no other than a binary system can, with convenience, be used. In fact, in the subdivisions of the unit, no other method appears to be consistent with nature; and those systems which are founded on division by two, appear to defy any attempts to alter them. Thus the system of money in the United States, which is strictly decimal, is only used in written calculations; while the old binary division of the Spanish dollar is retained in all retail operations, in spite of the barbarous nomenclature that is applied to it in some of the States. Several of the units of the French system, or their decimal divisions and multiples, are unsuited to ordinary transactions; subdivisions suitable to these were, therefore, first introduced clandestinely, and afterwards sanctioned by law. Thus a measure of the length of about a foot, is the most convenient for many mechanical uses; and for this purpose a measure of the third part of a metre was formed, called the Metrical Foot, to which the ancient duodecimal subdivision was applied. The kilogramme differing but little from two ancient pounds, its half has become the unit of weight in actual use, and is called the Metrical Pound; to this, also, a binary division has been applied, and the decimal system in the desending scale has not only failed in being introduced into commerce, but has been abolished by authority.

Thus there are at present in France, in fact, three diverse systems; the ancient, which is not wholly abandoned; the decimal system of the commission; and a system derived from the latter, to which the ancient names, and many of the ancient subdivisions are applied.

The system proposed, and partially introduced by the French philosophers, may, therefore, be considered as a splendid failure, worthy however of a better fate, from the scientific skill with which the operations connected with it were executed. It is also memorable for the light it has thrown on all analogous processes, and the actual benefit the researches, in respect to it, have conferred upon physical science. Warned by the example of the French, the British, Danish, and American governments, have wisely restricted themselves to the verification of the measures in actual use, and their restoration to their true dimensions. The two former, and the state government of New-York, have referred them to the pendulum, a standard existing in nature, determinate, and easily determinable.

The determination of units of measures of capacity, and of weight, from standards existing in nature, involves certain nice-

ties founded on the mechanics of fluid bodies. These will be fully explained in a subsequent part of this work.

297. Among the applications of the theory of the pendulum, may also be classed the principle of the calculations by means of which the density of the earth is ascertained from the experiment of Cavendish, § 91. The balls attached to the balance being set in motion, and caused to oscillate by the attraction of the masses of lead presented to them, may evidently be considered as a horizontal pendulum actuated by that attractive force. By comparing the length of this pendulum with that of a common pendulum, that would oscillate in the same time, we may obtain the relation, between the attractive force of the spheres of lead, and of the earth, considered as a sphere. The equation that expresses this relation may be thus investigated:

Let us consider the bodies attached to the extremities of the horizontal bar of Cavendish's apparatus, as if their masses were collected in a single point, and abstract the mass of the bar itself, so that each arm of the balance may be considered as a simple pendulum.

Let the length of the arm $=a$; the distance of the centre of gravity of the attracting mass, from the point of suspension of the bar, $=c$; the angular distance between the end of the bar, and the centre of gravity of the attracting mass, at the time motion begins, $=\alpha$; their mean angular distance in the oscillations $=\beta$; the measure of the attractive force exerted at the distance of the unit of lineal measure in which the distances are estimated, by a mass whose magnitude is equal to the unit of weight in which the masses are estimated, $=f$; the distance between the attracting mass, and the end of the bar at the time motion begins, $=c$.

Call the measure of the attractive force of the earth, g, the radius of the earth, R, and its mass m, we shall have for the value of g, in terms of m, f, and R,

$$g=\frac{mf}{R^2}. \qquad (a)$$

We should in like manner have for the value of the attractive force of the mass employed in the experiment, provided its distance from the end of the pendulum were constant, in terms of its mass m', of the force f, and the distance c,

$$\frac{m'f}{c^2}.$$

But this force does not act directly: it must, therefore, be decomposed into two, one of which is perpendicular, the other parallel to the bar that oscillates. The former alone acts, and its value will be from § 13,

$$\frac{m'fa \sin. \alpha}{c^3}.$$

We are next to consider that this is not a constant force, but that it acts with the least intensity at the time motion begins, and increases until the bar approaches nearest to the attracting mass. At this point, the torsion of the wire of suspension overcomes the motion, and causes the bar to return. When the deflection $=\beta$, the two forces balance each other, and at this time we have the mean value of the attractive force which will therefore be a function of β. And we may without *investigation assume, what might be shown by a rigorous analysis, that it is inversely proportioned to β, or that to find the value of the attractive force that acts on the balance at its mean intensity, the foregoing expression must be multiplied by $\frac{1}{\beta}$. Hence we have for the value of the attractive force g', that acts to cause the oscillations,

$$g'=\frac{m'fa\sin.\alpha}{c^3\beta}. \qquad (b)$$

The general expression, (186), gives for the value of the time of the oscillations of any pendulum under the action of an attractive force, g,

$$T=\pi\sqrt{\frac{l}{g}}.$$

If we call the length of the common pendulum, whose oscillations are synchronous with those of the bar l, we have by substituting the value of g, from (a), and squaring

$$T^2=\frac{\pi^2R^2l}{mf};$$

and by substituting the value of g', from (b), and also squaring

$$T^2=\frac{\pi^2c^3\beta b}{m'fa\sin.\alpha};$$

hence

$$\frac{\pi^2R^2l}{mf}=\frac{\pi^2c^3\beta b}{m'fa\sin.\alpha},$$

or

$$\frac{lR^2}{m}=\frac{c^3\beta b}{m'a\sin.\alpha};$$

whence we obtain

$$\frac{m}{m'}=\frac{lR^2a\sin.\alpha}{c^3b\beta};$$

and all the quantities in the second number of the expression may be obtained from experiment; and thus the ratio between the mass of the earth, and that used in the experiment, will admit of calculation.

* See Poisson, vol. ii. p. 42.

CHAPTER VI.

Of Collision.

298. The simplest mode in which motion can be communicated from one body to another is by collision. It is unnecessary for us to inquire whether actual contact takes place in this case between the particles of which bodies are composed. It is sufficient for us to know that the result of all experiment is precisely such as would happen, were the contact actually to occur.

All bodies in nature are more or less compressible, and when they have been compressed, tend in a greater or less degree, to recover their original figure. In some, this tendency is extremely small; in others it is considerable; it is styled their Elasticity. When bodies restore themselves to their original figure, after being compressed, and their particles in restoring the figure, return with a force equal to that by which the compression was effected, they are said to be perfectly elastic. If they did not yield to compression, they would be hard, and wholly devoid of elasticity. Of the latter class, there are probably no bodies in nature; but there are some that retain any figure that may be impressed upon them, having little or no tendency to restore themselves to their original shape.

Gases and vapours are, within certain limits, perfectly elastic; bodies of other classes differ materially in this respect.

299. In investigating the laws of collision, we consider bodies either as perfectly elastic, or as wholly devoid of elasticity; and may thence finally conclude what would occur in the case of imperfect elasticity. The simplest law is that which governs the collision of non elastic bodies; from this, too, the circumstances of the collision of perfectly elastic bodies, may be directly deduced. The former of these must, in consequence, be first investigated:

Let A and B be two non-elastic bodies, homogeneous, and of a spherical figure; and let their centres move in the same straight line, in such a manner, that, when they strike, the point of impact shall be in the line that joins their centres. Let a and b, be their respective velocities, which if in the same direction, will have the same, if in contrary directions, opposite algebraic signs. When they strike against each other, each will yield to a greater or less extent, until the whole action have taken place, and the time in which this occurs may be considered as inappreciably small; in all cases it is in fact insensible. So soon as the whole action has taken place, the two bodies will move forward with equal velocity,

for their is no force to act, provided they be non-elastic, to separate them. Call this common velocity, v, the quantity of velocity lost or gained by A, will be $a-v$; and that lost or gained by B, will be $b-v$; and these will be positive, or negative quantities, according to the conditions of the problem.

The quantities of motion the bodies respectively lose or gain, will be

$$Aa-Av, \text{ and } Bb-Bv;$$

hence by the principle of D'Alembert, § 69,

$$(Aa-Av)+(Bb-Bv)=0;$$

whence we obtain for the value of v,

$$v=\frac{Aa+Bb}{A+B}. \tag{321}$$

The whole quantity of motion will be,

$$Av+Bv; \tag{322}$$

and the respective quantities are, Av, and Bv.

If one of the bodies, A, be alone in motion, and B at rest, $b=0$ and

$$v=\frac{Aa}{A+B}. \tag{323}$$

If the bodies move in opposite directions, a and b will have contrary signs; and if a be positive,

$$v=\frac{Aa-Bb}{A+B}. \tag{324}$$

If the bodies be equal, and the velocities equal, but in contrary directions,

$$v=0. \tag{325}$$

If the bodies have equal quantities of motion in opposite directions

$$Aa=Bb, \tag{326}$$

and

$$A:B::b:a;$$

in which case we again have

$$v=0. \tag{327}$$

To express these formulæ in words:

When two non-elastic homogeneous bodies of a spherical form impinge against each other, at a point situated in the line that joins their centres, if the sum of their quantities of motion before impact, be divided by the sum of their masses, the quotient is the common velocity after impact. The quantities of motion must be considered as having the same or contrary signs, according as the motions are in the same or contrary directions.

If one of the bodies is at rest, its quantity of motion is to be divided by the sum of the masses.

If the bodies be equal, and move in opposite directions with equal velocities, they destroy each other's motions, and both come to rest. The same is the consequence if unequal bodies meet, with equal quantities of motion, or which is the same thing, when their velocities in opposite directions are inversely as their masses.

It will be also seen from inspection of the formula (321), that the change of motion that takes place in each of the bodies is equal, and that the change in their velocities is inversely as their masses.

The sum of their motions, estimated in the same direction, is the same before and after impact, and the state of their centre of gravity, whether it be at rest or in motion, is not changed by their mutual action.

If one of the bodies be infinitely large in respect to the other, and at rest, then the common velocity becomes infinitely small, and no error can arise in taking

$$v=0.$$

Such is the case in all obstacles that are considered as immoveable, which are so only in consequence of their great magnitude, or from their being firmly connected with the surface of the earth. If their surfaces be plane, they may be considered as portions of spheres of infinite magnitude. Hence, when a non-elastic body strikes against a plane surface in the direction of a normal to it, it will be brought to rest upon it; and so, as we may consider curved surfaces, when the impact takes place at a single point, as made up of planes, rest will again take place whenever a spherical and homogeneous non-elastic body strikes in the direction of a normal to the surface, against any immoveable obstacle.

300. If a spherical body strike against an immoveable plane surface in any other direction than that of a normal, resolve its moving force into two components, one of which is in the direction of a normal to the surface, and the other coincides with the surface. That part which is in the direction of the normal, will be destroyed by the resistance of the surface; the other part will remain acting upon the body, which will, therefore, move along the surface under its influence; and the new direction will be defined by its being in the plane, passing through the original direction of the moving body.

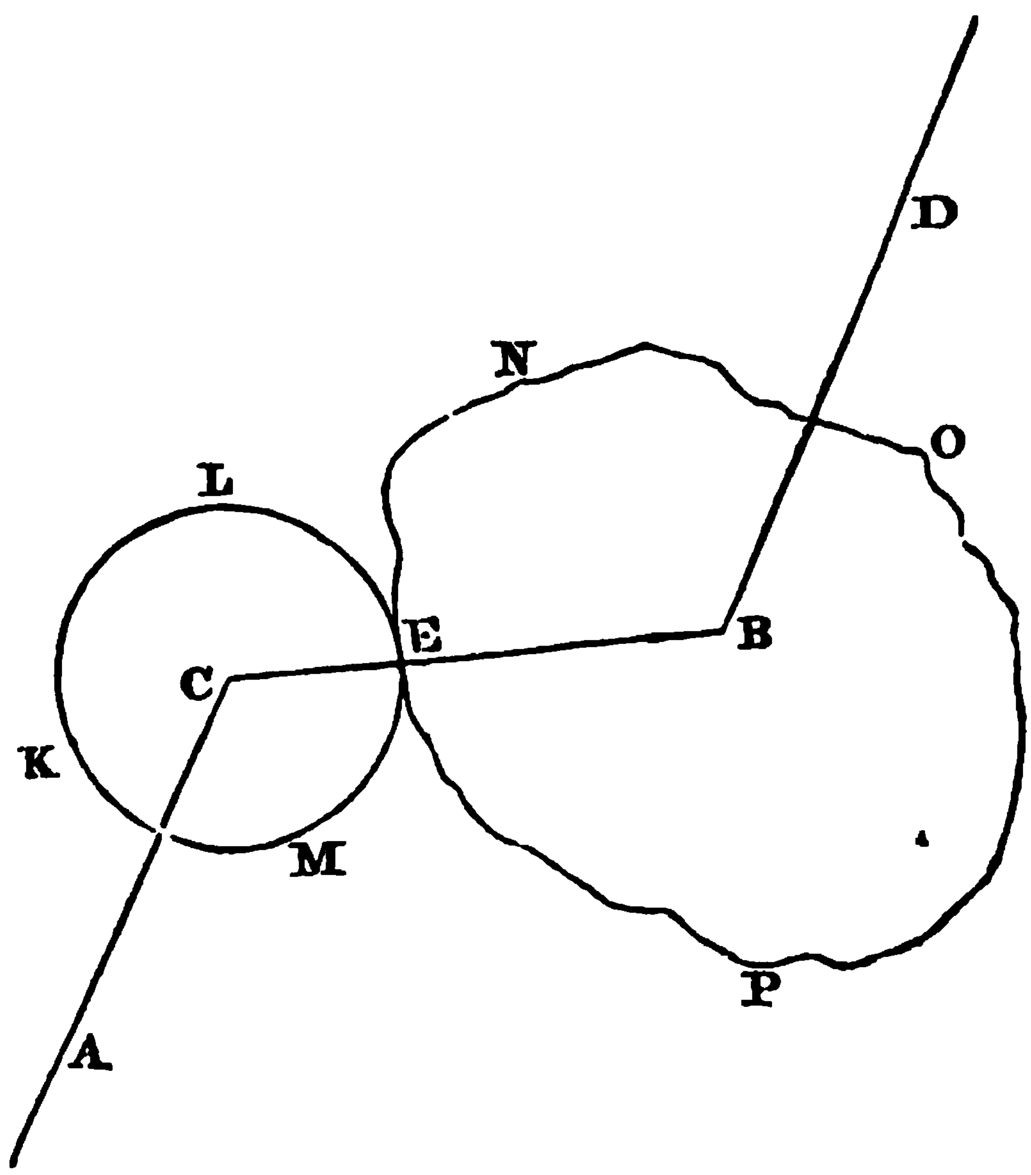
301. If the two spherical bodies do not move in the same straight line, but have the directions of their motions inclined to each other, the bodies will go on together in the direction of the resultant of their respective quantities of motion, and the sum of their new motions will be represented in magnitude by this resultant.

The determination of the direction and quantity of motion in this case may be effected by a simple geometric construction, which is as follows: Draw from the point of concourse lines to represent the magnitudes and directions of the forces whose intensities are the respective products of the masses of the two bodies into their velocities; call these lines A and B. Resolve one of these, B, by the construction of a right angled parallelogram into two others, one of which, *a*, is parallel, the other, *b*, perpendicular to A: produce A on the other side of the point of concourse, until the length of the produced part$=A+a$; produce B in like manner until the part produced$=b$: on the two lines produced, $A+a$ and *b*, construct a parallelogram which will be rightangled, and draw the diagonal. The sum of the motions of the two bodies after impact, will be represented in magnitude and direction by this diagonal; and if it be divided into two parts, proportioned to the masses of the two bodies, these will represent their respective quantities of motion.

302. If two bodies, moving in parallel lines, strike against each other, the circumstances of their motion, after impact, will differ from the case we have considered, in which their centres of gravity move in the same straight line.

Suppose, first, that the bodies strike in such a manner that the direction of one of them shall be a common normal to their surfaces.

Let KLM, NOP, be sections of the two bodies, in a plane



passing through their respective centres of gravity, and the line AC, which represents the direction of A; this direction will have for its distance, x, from the centre of gravity of B, the line BC. Let a' be the velocity that remains to the body A, after impact. The mass B being acted upon by a force whose direction does not pass through its centre of gravity, will derive from it two motions, one of which is progressive, the other rotary, § 245, around an axis passing through its centre of gravity, and perpendicular to the plane assumed for the section of the body. Let u be the velocity of the progressive motion, ω, the angular velocity of the rotary motion. Let a and b represent the respective motions of A and B before impact.

The body A will gain or lose a velocity represented by $a—a'$ the body B, a velocity represented by $b—u$, and by the principle of D'Alembert,

$$A.(a-a')+B(b-u)=0. \qquad (328)$$

The body, B, will revolve with a force equal to that which A loses by the collision. If then the moment of inertia of the body, B, in respect to the axis passing through its centre of gravity, $=S$, we shall have (241) for the value of the angular velocity,

$$\omega=\frac{Ax(a-a')}{S}. \qquad (329)$$

For the same reason that spherical bodies, § 297, do not separate after impact, the body, A, will go on with a common velocity with the point C, of the body B. This velocity is $u—x\omega$, hence,

$$a'=u-x\omega. \qquad (330)$$

If the bodies do not strike in a point that is in the common normal to their two surfaces, both will be caused to revolve, and the rotary motion of A, may be estimated upon the same principles.

303. When the bodies that impinge against each other are elastic, they will no longer go on together with a common velocity, but will be separated by the action of their elasticities. If the elasticity be perfect, each body will tend to restore itself to its original form, with a force equal to that by which the change of figure has been produced; this force will be equal to the change of motion it would undergo were it non-elastic. Whatever quantity of motion, then, a body loses or gains, when non-elastic, will be exactly doubled if it be perfectly elastic. On this principle, we may proceed to investigate the laws of the collision of perfectly elastic bodies.

Supposing the bodies to be spherical, and to move in the same straight line, using the same notation as in § 297; their common velocity, if non-elastic would be (321)

$$v=\frac{Aa+Bb}{A+B};$$

the body A, would, in this case, lose or gain a velocity represented by

$$v-a,$$

and it will lose or gain as much more on account of its perfect elasticity; hence its actual velocity will be

$$a'=2v-a; \tag{331}$$

substituting the value of v, we have

$$a'=2\left(\frac{Aa-Bb}{A+B}\right)-a,$$

whence

$$a'=\frac{Aa-Bb+2Bb}{A+B}, \tag{332}$$

or

$$a'=\frac{(A-B)\,a+2\,Bb}{A+B}; \tag{333}$$

pursuing the same course of reasoning in respect to B, we obtain

$$b'=\frac{Bb-Ab+2\,Aa}{A+B}, \tag{334}$$

or

$$b'=\frac{(B-A)\,b+2\,Aa}{A+B}; \tag{335}$$

when the bodies are equal,

$$a'=b, \text{ and } b'=a. \tag{336}$$

The relative velocity before impact, is $a-b$, after impact, $a'-b'$; by subtraction of the two last, we obtain

$$a-b=-a'+b', \tag{337}$$

and

$$a+a'=b+b', \tag{338}$$

whence we have for the sums of their motions before and after impact,

$$Aa+Bb=Aa'+Bb'. \tag{339}$$

Hence we may infer, that—

304. When two perfectly elastic and homogeneous spherical bodies, moving in the same straight line, impinge against each other, they separate after impact, and move each with a different velocity; if we estimate the velocities that would be lost or gained by each, if non-elastic, and add or subtract this, to or from, the common velocity they would have upon the same hypothesis, we obtain their actual velocities after impact.

If the masses of the bodies be equal, their velocities are interchanged. Thus, if one only of the bodies be in motion, it will come to rest after impact, and the other will move forward with the original velocity of the first; if they move in opposite direc-

tions, each will return in the direction whence it came, with the velocity the other had: if they move in the same direction with unequal velocities, the more swift overtaking the more slow, that which moved most rapidly will move with the less velocity the other had before impact, and *vice versa*. Whether the bodies be equal or unequal, the difference in their velocities, or what is styled their Relative Velocity, will be the same in amount both before and after impact; but they will have contrary signs.

When an elastic body strikes against another that is at rest, and larger than itself, the former will return in the direction in which it was proceeding before impact; and if the latter be infinitely large when compared with the former, the motion of return or reflection will have the same velocity with that of impact. For the same reason, a body perfectly elastic will be reflected back by a plain surface, in a direction opposite to that in which it came, and with a velocity equal to that it originally had.

305. If the impact be oblique, the reflection will not take place in the same straight line, but will still occur; and it may be inferred, from an application of the principle of elasticity to § 298, that the directions of reflection and incidence will both be situated in the same plane perpendicular to the surface of impact, and that the angles of incidence and reflection will be equal to each other.

306. There are several cases where, when we have occasion to estimate mechanical effect, it becomes necessary to consider the product of the mass by the square of the velocity. This product is called the *Vis Viva* of the body, in reference to which it is estimated. When non-elastic bodies inpinge against each other, the sum of these products of the two bodies does not remain constant; but it is otherwise with bodies that are perfectly elastic.

If we compare the sums of the forces estimated in the above manner in the case of non elastic bodies, it will be found that they have diminished; in the case of perfectly elastic bodies, these products remain constant, as may be readily shown.

Taking the formula, (339),

$$Aa+Bb=Aa'+Bb',$$

and

$$A(a-a')=B(b-b');$$

then by (338),

$$a+a'=b+b';$$

multiplying these two equations, we have

$$A(a^2-a'^2)=B(b'^2-b^2);$$

whence we obtain

$$Aa^2+Bb^2=Aa'^2+Bb'^2. \quad (340)$$

307. When a small elastic body in motion strikes against a greater elastic body at rest, it has been stated that the former would be reflected; the larger will be set in motion, and will proceed forward in the direction in which the smaller was proceeding before impact. This will be readily seen by considering, that were the bodies not elastic, the whole motion would be distributed among them in proportion to their masses; hence, when we add as much more motion to that the larger body would receive under this hypothesis, the sum will be more than the original quantity of motion in the smaller body. The whole quantity of motion in the two bodies is not on this account increased, for the smaller body is reflected with a quantity of motion exactly equal to the increased amount communicated to the greater body, and the algebraic sum of the motions estimated in one direction is constant.

When several elastic bodies, increasing in magnitude, are arranged in the same line, touching each other, and one smaller than the least of them strikes against it, each will in turn be reflected, and communicate to the succeeding one a greater quantity of motion than itself has. This increase of motion is greatest when the masses of the bodies increase in geometric progression.

Let there be three elastic homogeneous bodies of spherical figure, A, B and C, of which A is in motion, and strikes against B, in a line passing through the centres of the three bodies; the velocity communicated to B will be, (334),

$$\frac{2Aa}{A+B}; \tag{341}$$

and to C, calculated also from (334),

$$\frac{4ABa}{(A+B)\times(B+C)}=\frac{4Aa}{(\frac{A}{B}+1)\,(B+C)}$$
$$=\frac{4Aa}{A+B+\frac{AC}{B}+C}; \tag{342}$$

if the magnitudes of A and C are given, the last expression will obviously be a maximum, when

$$B^2=AC,$$

or when A, B and C, are in geometric progression, and the same may be proved of any number of elastic bodies whatsoever.

308. When the bodies before collision move in parallel lines, or when they meet in lines making an angle with each other, the principles of § 299 may be made use of to estimate the quantities of motion that would be lost or gained by each, if non elas-

tic, and the change that is produced in the direction of the motion. Knowing, then, that both the change in the direction, and in the quantity of motion, is doubled by perfect elasticity, the manner in which the bodies will move in the first case, and the direction and intensity in the second, subsequent to collision, may be investigated. In these cases, the general formulæ of the composition and resolution of forces will meet every possible instance.

309. If we examine into the state of the centre of gravity, we shall find that whether it be in rest or in motion before the collision of the bodies, that state of rest or motion will not be affected by their impact; the same has been found to be the case in the collision of non-elastic bodies; and this is a constant law in whatever manner the bodies act upon each other, and whatever be their respective natures.

310. When the elasticity of bodies that impinge against each other is not perfect, the quantity of motion lost or gained by each, in consequence of their elasticities, will be less than in the case of perfect elasticity.

Let p be the relation of the force of elasticity to the compressing force, the formulæ, (333) and (335), will become

$$a' = a - (1+p)\frac{B\,(a-b)}{A+B},$$

and

$$b' = b - (1+p)\frac{A\,(a-b)}{A+B}.$$

If we estimate the loss of *vis viva*, which takes place in consequence of collision, we shall find it to be

$$(1-p^2)\frac{A\,B(a-b)^2}{A+B}; \tag{343}$$

when the bodies are perfectly elastic, $p=1$, and this expression $=0$; and when non-elastic, $p=0$, and the expression is a maximum.

BOOK V.

OF THE EQUILIBRIUM OF FLUIDS.

CHAPTER I.

General Characters of Fluid Bodies.

311. Fluids are distinguished from solid bodies, § 79, by the greater degree of ease with which their particles may be separated. It is the distinctive and characteristic property of fluids, that these particles may be moved among each other by the smallest possible force. The physical agent that causes fluidity is heat, existing in a latent state.

When the repulsive force, exerted by heat, exactly balances the attraction of aggregation that exists among the particles, the body is a liquid; when the force of heat preponderates, the body has a tendency to expand itself; this tendency is opposed, and may be overcome by pressure, and hence the body will occupy spaces of different extents, that will depend upon the intensity of the compressing force: such a fluid is said to be elastic.

We know of no perfect liquids in nature; in them all the force of the attraction of aggregation slightly exceeds the repulsive force of heat, as is manifested by small portions of liquids arranging themselves in the form of spherical drops, and by their particles resisting forces that tend to separate them, although with a feeble intensity; this resistance is called Viscidity.

312. Liquids, in examining the theory of their equilibrium and motion, are considered as incompressible, and consequently wholly devoid of elasticity. A want of this property was, for a long time, considered to be essential to liquids. It might, however, have been inferred, that as liquids are capable of contracting and expanding with changes of temperature, they were not absolutely incompressible; and that a mechanical agent of intensity equal to heat, might cause them to change their bulk.

This inference has been confirmed by the experiments of Canton, and more recently by those of Perkins and Oersted, who have shown, that the bulk of water may be diminished by pressure. This diminution in bulk is about .000048 for a pressure equivalent to 15 lbs. upon a square inch of surface; an amount that, as will be shown hereafter, constitutes that unit of pressure which is called an Atmosphere.

313. To determine the nature and intensity of the forces that act upon any given particle of a fluid mass, we may, in consequence of the very small size of the particles, and the apparent continuity of the mass, ascribe to these particles any figure that we think proper, or which is most convenient for the purposes of our investigation.

If we take then a given particle of the fluid, and refer its position to three rectangular co-ordinates, X, Y and Z, we may consider the particle as occupying a space extended in three dimensions, and having the figure of a parallelopiped, whose dimensions are dx, dy and dz; and may assume for its volume the product of its three dimensions. If we call its density s, its mass will be

$$s\,dx\,dy\,dz. \tag{344}$$

The forces that act upon this particle, must be either inherent in the particle itself, and intrinsic; or must be exercised in the form of pressure by the surrounding particles, and therefore extrinsic.

The intrinsic forces may, § 17, be resolved into three, acting at right angles to each other, in the directions of the three axes, X, Y, Z; if we call these components P, Q and R, the forces that solicit the particle will be found by multiplying these components respectively by its mass, or they will be

$$\left.\begin{array}{l} Ps\,dx\,dy\,dz, \\ Qs\,dx\,dy\,dz, \\ Rs\,dx\,dy\,dz. \end{array}\right\} \tag{345}$$

The extrinsic forces are pressures, and may be represented in each case by taking the surface dk, on which the pressure is exerted, and multiplying it by the height of a column of homogeneous matter, whose weight is equal to this pressure, and by the density of this column. If we suppose the density of the column to be 1, and call its height p, the pressure will be

$$p\,dk. \tag{346}$$

This pressure must be exercised in the direction of a normal to the surface dk: for were the force oblique; of its two rectangular components, one in the direction of the surface itself, the other in that of its normal; the latter alone could produce any effect. And it is therefore obvious, that all the forces that can act upon a given surface of a fluid particle, must be finally reducible to two, act-

ing in opposition to each other, in the direction of its normal; these forces may be called

$$p\,dk, \text{ and } p'dk. \tag{347}$$

And in the case of equilibrium, these pressures must be equal, and

$$p=p'. \tag{348}$$

If the surface pressed change its direction, without having its position in the mass of fluid changed, the value of p will remain constant. For: suppose the surface to turn around one of its edges, until it make with its original position the angle i: The magnitude of the surface that is pressed by the column, the latter remaining unchanged in area and volume, will be

$$\frac{dk}{\cos. i};$$

the length of the perpendicular height of the column in its new position, from the surface, will be

$$p \cos. i;$$

the product of these two quantities is the amount of the pressure, or

$$\frac{dk}{\cos. i} \cdot p \cos. i = dk\,p.$$

But in different parts of the same mass, the quantity p will vary, and will depend upon, or be a function of X, Y and Z.

If then we take another surface contiguous to the first, the column that measures the pressure upon it will have for its altitude,

$$p \mp dp,$$

and the pressure will be, if the surface have the same area as the first,

$$dk(p \mp dp).$$

To apply these principles to the case of the elementary parallelopiped: p is, as has been stated, a function of the three co-ordinates; we may, therefore, assume for the value of its differential,

$$dp = \mathrm{L}\,dx + \mathrm{M}\,dy + \mathrm{N}\,dz;$$

and the three co-efficients, L, M and N, will be the differentials of p, in the respective directions of X, Y and Z.

The face whose surface is $dy\,dz$ will be pressed in the direction of a normal, which direction will be the same as that of x, by a force represented by

$$p\,dy\,dz;$$

the opposite and equal face of the parallelopiped will be pressed by a force in the opposite direction, whose measure is a column or prism, whose base is $dy\,dz$, and altitude $p+\mathrm{L}\,dx$; this force will therefore be

$$(p+\mathrm{L}\,dx)dy\,dz;$$

and upon the remaining four surfaces, the pressures will be

$$p\,dy\,dx,$$
$$(p+Nz)dy\,dx,$$
$$p\,dz\,dx,$$
$$(p+M\,dy)dz\,dx.$$

These six forces, acting in opposite directions by pairs, may be resolved into three, which are

$$\left.\begin{array}{l} L\,dx\,dy\,dz, \\ M\,dx\,dy\,dz, \\ N\,dx\,dy\,dz. \end{array}\right\} \qquad (349)$$

These three rectangular forces will be the resultants of all the extrinsic forces that act upon the particle. If with these be combined the rectangular resultants of the intrinsic forces, (345), we have, for the rectangular resultants of all the forces that act,

$$\left.\begin{array}{l} (Ps-L)dx\,dy\,dz, \\ (Qs-M)dx\,dy\,dz, \\ (Rs-N)dx\,dy\,dz. \end{array}\right\} \qquad (350)$$

314. The nature and intensity of the forces that act upon any given particle of the fluid, being obtained in the preceding expressions, we may proceed to investigate the conditions of equilibrium that exist among them.

As every particle of a fluid mass has, by the definition of that class of bodies, perfect freedom of motion, except such resistances as may arise from the pressure of the surrounding particles; every particle must in consequence, when the mass is in equilibrio, be also in equilibrio under the forces that act upon it. The three rectangular resultants of the forces, (350), must therefore be respectively $=0$. Hence, in the expressions, (350), we have

$$L=Ps,\ M=Qs,\ N=Rs;$$

if we multiply these equations respectively by dx, dy, dz, and add the products together, we obtain

$$dp=s(P\,dx+Q\,dy+R\,dz), \qquad (351)$$

which is the equation that gives the conditions of equilibrium.

315. The resistance to the motion of the particles of fluids, is restricted to the pressure of the surrounding mass; while in solids, not only does this resistance act, but one of far greater intensity, namely, the attraction of aggregation that exists among their particles. The above condition of equilibrium therefore exists in solids as well as in fluids; that is to say, that if it obtain, the mass will be in equilibrio, but we cannot infer, that if it do not obtain, the mass will therefore cease to be in equilibrio.

It results from the nature of fluid bodies, whose particles are free to move, that if any one of them be set in motion, all the

rest must be set in motion also; and that the application of any force to any one of the particles, in addition to those whose resultants are given in the equation (351), will affect the equilibrium of all the rest; therefore a pressure applied to any point of a fluid mass, will be propagated in all directions, and influence every particle of which the mass is made up. A pressure applied to any point of a solid body, would, in like manner, be propagated in all directions, were it not that it is counteracted by the attraction of aggregation. Hence, the property usually assumed as the basis of the mechanics of fluid bodies, namely, that of propagating pressure equally in all directions, is not distinctive; and solids, in which the attraction of aggregation is weak, or in which it has been overcome by mechanical action, will press to a certain extent in conformity with the laws that govern the pressure of fluids. This is found to be the case in masses of sand and loose earth, that often produce mechanical effects similar, although not absolutely identical with those produced by fluids. In consequence of this mode of action, the investigation in § 196, of the strength of a wall that will support the pressure of earth, does not always give sure results.

CHAPTER II.

Of the Equilibrium of Gravitating Liquids.

316. All fluids upon or near the surface of the earth, are influenced by the attraction of gravitation ; in liquids thus situated, no other cause exists for the mutual action of their particles upon each other. This force is always exerted in the direction of a line tending to the centre of the earth, at which point all the directions of the force of gravity at the surface would meet, were the earth a perfect sphere. The direction then, at any place, is a vertical line.

If we suppose that this direction coincides with the axis z, the quantities P, and Q, in the direction of the other axes, become in liquids$=0$. R, alone remains, and if we consider this as equal to unity, the equation (351) becomes

$$dp=s\,dz\,; \tag{352}$$

and integrating,

$$p=s\,(A+Z)\,;$$

the force p, which measures the pressure upon any point of a gravitating liquid, will, therefore, be a function of the variable quantity z, or will depend upon its vertical depth beneath the surface of the fluid. Hence, every point situated at the same distance below the surface of a mass of fluid, will undergo an equal pressure.

At the surface of a gravitating fluid, the quantity dp, which depends upon the depth of the fluid, is $=0$; hence, the equation (351) of the surface, when in equilibrio, becomes

$$s(P\,dx+Q\,dy+R\,dz)=0\,;$$

if we take the density $s=1$,

$$P\,dx+Q\,dy+R\,dz=0\,; \tag{353}$$

and in the case of a liquid,

$$dz=0. \tag{354}$$

317. When a gravitating liquid is placed in an open vessel, its particles move in consequence of the fundamental property of fluids, until they reach a state in which the conditions of equilibrium are satisfied. It will then come to rest, and assume an uniform and constant surface. The last equation (354) shows that every point in this surface will be at an equal distance from the plane to which the axis z is perpendicular; the surface will, therefore, be plane, and at right angles to the direction of the force of gravity. The last part of the proposition is also true,

when the extent of the surface becomes so great that the directions of gravity can no longer be considered as parallel; but in this case, the surface itself becomes curved, and being every where at right angles to the force of gravity, is parallel to the mean surface of the terrestrial spheroid. Such a surface is said to be level, and is spontaneously assumed by all stagnant fluids upon the surface of the earth; and in masses that are agitated by extrinsic forces, as the ocean, which is in constant motion under the action of the winds, and the causes that produce the tides, the mean surface is level.

That the surface of a liquid, when in a state of equilibrium under the action of any force whatsoever, must be perpendicular to the direction of that force, may be shown in another manner. For if we assume that the surface has not this direction, the force that acts on any particle, may be resolved into two, one perpendicular, the other parallel to the surface; the latter then will not be counteracted by fluid pressure, and the particle will move under its influence, which is contrary to the hypothesis of equilibrium. This mode of considering the subject shows in addition, what does not appear from the equation of equilibrium, namely, that the force that causes a fluid to assume a definite surface, must be directed from without the mass towards its interior.

318. The art of levelling consists in drawing a line, which shall every where intersect the direction of gravity at right angles. Such a line is in fact a curve, although it may be considered as straight, within narrow limits. Our instruments do not furnish the means of drawing the level curve, and we are, therefore, compelled to content ourselves with obtaining tangents to it, at the several points of observation. If these tangents be of no great length, and if they be equally produced each way from the point of contact with the curve, they form a polygon, that may, without sensible error, be considered as identical with the curve itself.

The levelling instruments, in most frequent use, are:

The Water Level, the Spirit Level, and the Mason's Level.

319. The water level is formed of a tube of glass, bent twice at right angles. If a fluid be placed in this tube, its surfaces will rise in each branch to the same height above the mean surface, as will be shown in a succeeding section. If the sight be directed over these two surfaces, the line of sight will be of course a tangent to the level curve.

320. The essential part of a spirit level consists of a cylindrical tube of glass, containing a portion of spirits of wine that nearly fills it; this tube is hermetically sealed at each end,

The space in the tube that is not occupied by the spirit, contains air, which appears in the form of a bubble. The air being much lighter than the spirit, the bubble will always occupy the highest part of the tube; and when the tube is truly horizontal, the bubble will come to rest at a distance exactly equal from each end. Any deviation from a horizontal position, will be marked by an approach of the bubble to the end of the tube that is most elevated.

In order to render the indications of the level more precise, it has been improved by grinding the interior of the tube into the form of a portion of a circular ring of large radius; and the axis of the tube is in consequence no longer a straight line, but an arc of a circle. If the radius of this circle be known in some conventional unit of measure, and divisions of the same unit be marked upon the tube, the deviation of the bubble from the exact middle of the tube may be measured by these divisions, and hence the inclination of the chord or line that joins the extreme points of the axis, estimated.

This level is usually mounted upon a tripod stand, that bears two plates that are ground to fit each other; one of these is at rest, the other moves upon it, around a vertical axis, and carries with it the tube. The conditions on which accuracy depends in this part of the arrangement, are, that the axis of the plates' motion shall be truly vertical; and that the axis of the tube shall be parallel to the surfaces of the plates, and of course perpendicular to the vertical axis of motion.

Whether both of these conditions be fulfilled, may be ascertained by one and the same operation, and the instrument adjusted, if necessary. The direction of the axis of the motion of the plate, is capable of change in respect to the stand, by means of four screws, called the levelling screws of the instrument. Such, at least, is the more usual number, and to assume that number as adopted, will best suit the purpose of explanation; although three screws would probably afford a more convenient adjustment, as may be inferred from § 112. The tube being brought directly over two of the screws, they are moved in opposite directions until the bubble exactly occupies the middle of the tube; the axis of the tube is, therefore, level. In order to determine whether it be parallel to the plates, and whether the axis of the latter be vertical, the moveable plate is turned around its axis through the half of a complete revolution; the position of the levelling tube is, therefore, reversed. If in this new position, the bubble still occupy the middle of the tube, the adjustment is complete. If, however, the bubble have approached to either end, one half of its change of position is obviously owing to a want of parallelism in the motions of the plate and the tube; the other half, to the

axis not being vertical. In order to correct the latter error, the levelling screws are again moved, until the bubble have passed back through about half the space that marks its change of position. To enable us to remedy the former error, the glass tube that contains the spirits of wine is enclosed in another tube of brass, open at its upper surface, to permit the bubble to be seen, and the divisions of the scale to be read. This outer tube is attached to the bar that supports it, at one end, by a hinge, whose axis is horizontal, and at the other by a vertical screw. The bubble having been moved back by the means just described, through half the distance it before passed through, in consequence of the error in adjustment, is brought exactly to the middle of the tube by the last named screw. The instrument being then turned around until it reach its original position, the bubble is again inspected; if it now come to rest exactly in the middle of the tube, the adjustment is complete; if not, the preceding operations must be repeated, until the bubble stand exactly in the middle, in both positions of the tube.

In order to direct the vision to the signals placed at the stations whose relative level is to be determined, sights must be adapted to the tube; that which has superseded all others, is the telescope; and the level, instead of being mounted as we have assumed, for the sake of more easy illustration, upon a simple bar, is attached by the hinge and vertical screw to the tube of the telescope. The telescope itself is mounted upon the horizontal bar that turns around with the moveable plate. For this purpose, the latter is furnished with two vertical supports; these have the form of rods, divided at their upper end into two inclined branches, and thus having the form of the letter Y, whence this part of the apparatus derives its name. One of these Ys has a motion in a vertical direction, by means of a screw placed beneath it. The telescope has upon its tube two collars, at an appropriate distance from each other, that are accurately turned, and rest upon the Ys. It is obvious, that the axis of the telescope, or more properly, the line that joins the centres of the two collars, must be rendered parallel to the axis of the spirit level. To ascertain if this be the case, and to correct it if it be not: the previous adjustment having been completed, and the bubble brought exactly to the middle of the tube, the telescope is lifted from the Ys, and again replaced in an inverted position; that is to say, each of the collars is made to rest in the Y that had before borne the other. If in this inverted position the bubble still occupy the middle of the tube, this adjustment is complete; but if it do not, one half of the error is due to the position of the parallel plates, the other half to the position of the Ys. In order, therefore, to effect this adjustment, the bubble is brought back to

the middle of the tube, partly by the levelling screws of the instrument, and partly by the screw that has been described as giving a vertical motion to one of the Ys.

A telescope having a field of view of a definite extent, it is necessary that some line should be defined within it, to point the direction of the vision to the signals. This is effected in the levelling instrument, by means of the intersection of two wires, the one vertical, and the other horizontal. These are placed in the common focus of the two lenses of the telescope, and are, in consequence of one of the optical properties of that instrument, as distinctly visible as the signal. The line passing through the eye, and the intersection of these wires, is called the Line of Collimation. In order to adjust this; the telescope, after the instrument has received the previous adjustment, is directed towards a moveable vane, on which a horizontal line is drawn; the vane is raised or lowered, until this line apparently coincides with the horizontal wire, in the focus of the telescope. The telescope is next turned half round, in the Ys, on its own horizontal axis; if the line on the vane correspond exactly with the horizontal wire, the instrument requires no farther adjustment; but if these lines do not coincide, half the error is evidently due to the position of the vane; the other half to that of the horizontal wire in the focus of the telescope. The correction is again double; the vane is moved through half the space that intervenes between the upper and lower visible position of the telescope; and the horizontal wire of the telescope, is moved through the other half, by means of screws, adapted for the purpose, to the tube of the telescope.

We determine in these adjustments, and in the subsequent use of the instrument, the proper position of the bubble, by observing the place occupied by its two extremities; for the bubble has no direct mode of showing the position of its exact middle. The length of the bubble is constantly varying with changes of temperature, for the spirits of wine expand and contract, when heated or cooled, and the air being elastic, accommodates itself to this change of bulk. Exposure to the sun affects the temperature, and his rays sometimes fall obliquely and partially. In the latter case, the indications of the spirit level, are uncertain, and the bubble will move from its position, when the instrument itself is perfectly at rest.

321. In using the level, it may be placed at one of the points, whose difference of elevation is to be determined; when the instrument has been levelled by bringing the bubble to the middle of the tube, by the motion of the levelling screws, and it has been ascertained that the bubble will continue in that position while it

moves around the vertical axis, the telescope is directed to the other point; at this a staff is set up in a vertical position, on which a vane slides. The vane has a horizontal line marked upon it, and an assistant slides the vane upon the staff, until this horizontal line appears to coincide with the horizontal wire of the telescope. The difference between the height of the eye-piece of the instrument, and that of the horizontal line upon the vane measured from the points on which they respectively stand, is the difference of level.

This determination will require corrections, unless the distance between the points be small. It is also liable to error, from a want of accurate previous adjustment in the instrument, and from the bubble not being exactly in the middle of the tube. The latter may be allowed for, by noting the divisions that point out the position of the level, and thence calculating the effect of the inclination of the axis of the telescope.

All these sources of error may be in a great degree compensated by the second, and more usual method. In this, the instrument is placed as nearly as possible at an equal distance from each of the points, whose difference of level is to be ascertained. A staff and sliding vane is set up at each of these points, and the telescope directed to them in turn. The lines upon the vanes having been made to coincide with the horizontal wire of the telescope, the difference in the altitudes of these lines, above the points on which the staves rest, is the difference of the respective levels. If the place of observation is exactly equidistant from the two points, the sphericity of the earth will equally affect both vanes, and the apparent difference will be the same as the true. So also if the axis of the telescope be from any cause inclined, or the line of collimation do not coincide with it, the errors that arise will be equal in both directions.

When a line of considerable length is to be levelled, it should be divided into parts by stations at equal distances; from 180 to 200 feet is a convenient space for this purpose. Staves being set up at the two first stations, the levelling instrument is placed at an equal distance from each of them, and the observations made, as has been directed: the staff from the first station is then moved forward to the third; that at the second remaining stationary. The instrument is next moved forward to a point equidistant from the second and third station, and a new pair of observations made. In recording the observations, the heights of the vanes, as seen from each position of the instrument, are arranged in two columns; the heights observed by looking towards the first station, or backwards, being set in one column; those obtained by looking forwards in the other. The difference between the sums of the numbers in the columns, gives the difference of level of the extreme stations.

322. The mason's level is composed of a ruler, to which a plumb-line is adapted, a line being drawn to mark the proper position of the cord, and an opening made to receive the plummet. A second ruler is adapted to the lower part of the first, whose lower edge is at right angles to the line with which the cord of the plumb-line is made to coincide; when the latter is brought to its proper position, the lower edge of the ruler is of course level, being at right angles to the plumb-line, and consequently to the direction of gravity.

A modification of this instrument, that is useful in many cases, may be constructed by suspending the plumb-line from a point at the intersection of two bars of equal lengths, which, therefore, form sides of an isosceles triangle; the position of the plumb-line is determined, by making it coincide with the middle of a bar, that is parallel to the line that would constitute the base of the triangle. Levels of this last form are sometimes constructed of considerable size, the bars being six or seven feet in length. They may be used for laying out ditches and trenches, for the distribution of water for agricultural purposes, but are not sufficiently accurate for the purposes of engineers.

323. The correction for the sphericity of the earth, applied to the first method of observation described in the preceding section, is obtained by considering the horizontal distance as a mean proportional between the earth's diameter, and the distance of the point at which the tangent cuts the diameter that passes through the point, whose level is to be determined, from the surface of the earth. This hypothesis is no more than an approximation, but is sufficiently near the truth, to be employed on most occasions.

Upon this hypothesis, if a be the horizontal distance between the two points; D the earth's diameter; and h the height of the horizontal vane above the level of the point at which the instrument is placed, all estimated in the same unit of lineal measure,

$$h=\frac{a}{D^2}\cdot, \qquad (355)$$

324. Observations, made with a levelling instrument at one of the points, whose relative heights are to be determined, may also be affected by atmospheric refraction. This is not the case when the instrument is placed half way between them. The correction to be used in the former case is not properly an object of investigation in mechanics, as it is derived from the principles of experimental physics.

325. When a homogeneous gravitating liquid, instead of being contained in a single vessel, or collected in a great mass in hollow basins on the surface of the earth, is placed in bent tubes, or

in vessels communicating with each other at bottom; the surfaces of the liquid in the several vessels or branches, will all form portions of the same general surface of level; or will, within a small space, be equally elevated or depressed below the same horizontal plane. For:

If in the first place we suppose the liquid to be contained in a tube or vessel, with no more than two branches, we may consider it as divided into two columns, by an imaginary surface, where the branches communicate. In order that equilibrium shall exist, the pressures upon this surface, exerted by the two columns, must be equal. Now, the fluid pressure upon a surface is a function of the vertical co-ordinate z, § 314, or will depend upon the vertical depth of the surface pressed, beneath the horizontal surface of the liquid. This quantity, z, must therefore be equal, in the case of equilibrium, in both branches, and the respective surfaces must be on the same level. The same mode of investigation will apply to the case of three, or any number of branches or vessels, communicating beneath the surface of the fluid.

If two heterogeneous fluids, incapable of being mixed, be placed in the two branches of a bent tube, the pressure exerted by one of them on the surface of contact, will be by the integration of (352),

$$p=\int sdz\,; \qquad (356)$$

and if s' be the density of the second fluid, and z' its depth, in the case of equilibrium, we have in the same manner,

$$p=\int s'dz'\,. \qquad (357)$$

In the case of liquidity, when each of these fluids may be considered of uniform density throughout, we obtain by the integration of the second members of the above equations, both of which are equal to p of (346),

$$sz=s'z'\,;$$

and

$$z:z'::s':s\,; \qquad (358)$$

hence:

326. The heights to which columns of heterogeneous liquids rise above the common surface of contact, are inversely proportioned to their respective densities.

If two immiscible liquids be introduced into the same vase, the denser of the two will occupy the lower part of the vase; for the freedom of motion which the particles of each possess, will allow it to descend, by its superior gravity, through the rarer. Hence, if the above proposition be submitted to the test of experiment, the denser liquid must be first introduced into the tube; for if the rarer be first introduced, the denser will descend through it, separate it into two columns, and occupy the bend of the tube. When the denser liquid has been introduced into the bent tube, the rarer, being poured into one of the branches, will,

by its pressure, force down the level of the denser in that branch, while that in the other will be elevated; but the depression will never carry the surface of contact beyond the bend in the tube, which will, therefore, continue to be occupied by the denser fluid.

The surface of contact must be level, for the depths, z, and z', are constant quantities.

327. When a solid body is wholly immersed beneath the surface of a fluid, it occupies a space identical with what it did before immersion; for mere immersion does not alter its volume. The whole volume of the fluid mass, when it contains the solid, will, therefore, be increased as much as is equal to the volume of the solid.

As the volumes of bodies of equal weights are inversely as their densities, the weights of fluid, displaced by bodies of different densities wholly immersed, will also be inversely as their respective densities.

When a body is thus immersed, its own weight tends to cause it to descend in the direction of the force of gravity; but this tendency will be counteracted, in a greater or less degree, by the action of the fluid. If instead of a solid body, we consider the case of one of the elementary particles of the fluid, and ascribe to it, upon the principle of § 313, the figure there assumed for one of the particles of the fluid: the pressures of the fluid upon the four vertical faces of the parallelopiped, will exactly counterbalance each other; the pressure on the upper surface, will be measured by a column of the fluid, (346), whose area is that of this surface of the particle, and whose altitude, is the vertical depth of the same surface, beneath the level of the fluid (considered as homogeneous); this depth in the case of a liquid, or other homogeneous fluid, will be the actual depth. The downward pressure on the lower base will be the sum of the weight of the superincumbent column, and the weight of the solid particle itself, while the upward pressure on the same base will be measured by a column of the fluid, whose altitude is the vertical depth of this lower surface. As the solid particle identically replaces in bulk an equal volume of the liquid, the difference of these two pressures will be the difference of the weights of the two particles; this, if the densities of the two be equal, will be $=0$. The same will also be true, as will appear from the same section, § 313, whatever be the direction of the surfaces of the particle. Now, as the solid body is made up of its particles, what is true in respect to one of them in this instance, will be true of its whole mass, and the tendency of the solid body to descend, under the action of gravity, will be diminished by a force, whose measure is the weight of a volume of the fluid, equal to the bulk of the solid body. Hence, if the solid

have the same density as the fluid, it will remain at rest in any part of the latter in which it may be placed: if its density be greater, it will descend; and if its density be less than that of the liquid, the solid will rise; but the force with which it moves, in either case, will have for its measure the difference in the weights of equal volumes of the two substances.

It will therefore be seen, that when a solid body is immersed in a fluid, it appears to lose as much of its weight as is equal to the weight of a mass of fluid whose volume is the same with its own. This loss of weight was by some considered as actual, and not merely apparent. This, however, is not the case, for the tendency to descend under the action of the attraction of gravitation is not destroyed, but merely opposed; the weight of the body, which is the sum of the gravitating forces exerted upon its several particles, § 105, still remains; but these forces are opposed by others acting in a contrary direction, and their joint resultant is of course less than the weight of the body. To render this more clear, if a vase be taken that contains a liquid, and if a solid body be immersed in it; although the latter will appear to lose a portion of its weight, the joint weight of the vase, the liquid and the solid, will still be the same, as if the latter were weighed separately, and its weight added to the joint weight of the other two substances.

328. When the solid has less density than the liquid, the resultant of the two sets of forces will be negative, and the apparent loss of weight of the solid will be greater than its own weight: it will in consequence rise to the surface of the liquid. Having reached the surface, it will continue to rise, and elevate a portion of its volume above the surface, until equilibrium between the two sets of forces be restored. The downward pressure on the lower surface becomes in this case no more than the weight of the solid itself; the upward pressure, in order to be equal to this, must have for its measure columns of fluid, whose joint weight is equal to that of the solid; and the joint volume of these columns will therefore be equal to the volume of the portion of the solid immersed. Hence, when a solid body of less density than that of the liquid, has risen to, or is placed upon, the surface of the latter, it will float there, displacing as much of the liquid as is equal in weight to the whole weight of the solid body.

The force which acts downwards, being the weight of the solid body, has for its point of application the centre of gravity of the solid; the force that acts upwards is the resultant of a number of parallel forces exerted by an infinite number of ver-

tical columns of the liquid, which together make up a volume equal to that of the part of the solid that is immersed. The resultant of such a system of forces will have for its point of application the centre of parallel force; this being determined upon, the principles of § 105, will be the same with the centre of gravity of the part of the solid immersed. And as it is necessary, not only that two opposing forces in equilibrio should be equal in intensity, but that they should act in the same straight line, this condition is also necessary for the the existence in equilibrium; and,

A solid will float in equilibrio upon the surface of a liquid, when it has displaced a volume of the liquid whose weight is equal to its own, and when the centres of gravity of the whole solid and of the part immersed are situated in the same vertical line.

329. If the same solid body be placed in succession in two different liquids, whose densities are both greater than its own, it will float at the surface of both. The part of the solid immersed will displace equal weights of the two liquids, but the volumes displaced will be different, if their densities be not the same. Now, as the volumes of equal weights of different bodies are inversely as their respective densities, the parts of the same solid that are immersed in two different liquids, on whose surface it successively floats, are inversely as their respective densities.

If the floating body have the form of a prism, and float, in conformity with the second of the above conditions of equilibrium, with its axis in a vertical position; as the horizontal area of the part immersed will be constant, the body will be immersed to depths measured vertically along its axis, that are inversely proportioned to the densities of the liquid on which it floats.

CHAPTER III.

OF THE PRESSURE OF GRAVITATING LIQUIDS.

330. The pressure of gravitating liquids upon surfaces immersed in them, or upon the sides of the vessels that contain them, may be easily determined from the principles of the preceding chapter.

The equation (352), gives us for the value of p,

$$p = s(A + z);$$

and if the origin of the co-ordinate, z, be at the surface of a homogeneous liquid,

$$p = sz;$$

substituting this value in the expression (346), for the pressure, P, we have

$$dP = dksz.$$

The measure of the pressure upon an infinitely small surface, therefore, is the weight of a column of the liquid, whose base is equal to the surface pressed, and altitude equal to its depth beneath the surface of the fluid.

Upon a plane surface of determinate magnitude, lying in a horizontal position, the whole pressure is the sum of the partial pressures upon all its elementary portions, and will therefore be

$$P = ksz, \tag{359}$$

or will be measured by the weight of a prism of the liquid, whose base is equal to the surface pressed, and whose altitude is equal to the distance of the plane from the horizontal surface of the liquid. If the plane be inclined to the horizon, call the distances of its respective elements from the surface of the liquid, z', z'', z''', &c., the sum of the partial pressures will be

$$P = s \int dk(z' + z'' + z''' + \&c.)$$

But if we call the distance of the centre of gravity of the plane beneath the level surface of the liquid, Z,

$$\int dk(z' + z'' + z''') = kZ;$$

and

$$P = skZ; \tag{359a}$$

and this will be true of any surface, whether plane or curved.

These several expressions, applied to the case of a liquid, contained in a vessel, are wholly independent of the bulk of the liquid itself; for they include no other quantities than the area of the surface pressed, and the depth of the liquid. Hence:

331. The pressure of a liquid, upon a horizontal plane, is equal to the weight of a prism of the liquid whose section is equal to the area of the plane; and whose altitude is equal to the depth of the plane beneath the surface of the liquid.

The pressure of a liquid upon a surface that is not horizontal, is equal to the weight of a prism of the liquid, whose section is equal to the area of the surface pressed, and whose altitude is equal to the depth of the centre of gravity of that surface beneath the level surface of the liquid.

Upon equal and similarly situated surfaces, the pressures are as the depths of the liquid; and at equal depths, the pressures are as the surfaces.

Upon a given surface, the pressure depends wholly upon the depth of the liquid above its centre of gravity, and has no reference to the quantity of liquid; and thus, in vessels having equal bases, and in which the liquid stands at equal heights, the pressures on the bases are equal, however different may be the respective capacities of the vessels.

If the vessel be prismatic, and the base horizontal, the pressure on the base is exactly equal to the weight of the liquid it contains; but if the vessel be wider at top than at bottom, the pressure on the base is less than the weight of the liquid it contains; while if the vessel be narrower at top, the pressure on the base exceeds the whole weight of the liquid.

332. When a liquid is in equilibrio in a vase, its surface is, as has been shown, § 315, horizontal: on the sides of the vessel, equilibrium exists in consequence of the fluid pressure being exactly counteracted by the resistance of the solid boundary. This resistance may be represented by a force, whose direction is a normal to the surface: hence the liquid pressure must also act in the normal, and will always have a direction perpendicular to the point of the surface on which it acts. In addition to the pressure on the bottom of the vessel, the sides are also pressed by forces, estimated as has been stated above; and thus in a prismatic vessel, not only will the base sustain a pressure equal to the whole weight of the fluid, but the sides will also sustain pressures perpendicular to their several surfaces: the measure of these forces is a weight of a prism of the liquid, whose horizontal section is equal to the area pressed, and whose altitude is equal to the depth at which the centre of gravity of the sides lies beneath the horizontal surface of the liquid. Thus, in a cube, filled with liquid, the pressure on the base is equal to the whole weight of the liquid, and acts vertically in the same direction with the force of gravity. On each of the four vertical faces, the pressure is equal to half the weight of the liquid mass, and acts in a horizontal di-

rection. A liquid, therefore, acting on the sides and base of a vessel, both changes the intensity and the direction of the force which acts, namely, the weight of the mass itself impelled by the force of gravity in a vertical direction.

A liquid then, is, in fact, by the definition of § 131, a machine; and masses of liquid may, and frequently are, used to produce effects analogous to those produced by machines.

333. In a close vessel, to which a lateral tube is adapted, if both be filled with a liquid, the pressure on the sides, the top and the base, will depend upon the areas of these substances, and on the height of the liquid in the lateral tube. This will be true, whatever be the respective dimensions of the vessel and the tube adapted to it; and thus any quantity of liquid, however small, may be made to counterbalance any other quantity however great. This principle is usually styled the Hydrostatic Paradox.

When a liquid is contained in a close vessel, and a pressure is exerted upon it by means of a piston fitted tightly to an opening made in one of its sides, this pressure may be represented by the weight of a mass of fluid resting upon the piston as a base; the action of the piston will therefore produce the same effect upon the liquid mass as a column of liquid, whose area is equal to that of the piston, and whose altitude is such, that the weight of the prism of liquid that has this area and altitude, shall be equal to the pressure upon the piston; the pressure exerted by the piston will therefore be equally felt upon all the sides of the vessel, and will upon an area equal to that of the piston, be equal to the whole pressure the latter exerts.

334. These principles may be subjected to the test of experiment, which fully confirms them; and experiments are of importance in this department of mechanics, inasmuch as we know nothing of the nature of the particles of which fluids are composed, nor of the manner of their action upon each other; the hypothesis on which we have proceeded in the mathematical investigation of the conditions of equilibrium, and of the measure of pressure, is obviously inaccurate, in omitting the viscidity, and the compressibility of liquids. When, however, we find that the deductions from the hypothesis are exactly consistent with experiment; we infer that the neglect of these circumstances produces no sensible error. The experimental illustrations are as follow:

(1.) If water or any other liquid be contained in an open vessel, it assumes a horizontal surface, except so far as it is influenced near the sides, by a force that will be hereafter the object of investigation.

(2.) If two or more immiscible liquids be poured into the same vessel, they arrange themselves in the order of their respective densities, the denser liquids occupying the lowest places.

(3.) If a liquid be poured into a bent tube, or into vessels communicating at bottom, it rises in both branches of the tube, and in all the vessels, to a common level, whatever be the areas of the branches of the tube, or the cubic contents of the several vessels.

(4.) Two heterogeneous liquids, placed in opposite branches of a bent tube, rise to heights above their surface of contact inversely proportioned to their respective densities; the denser liquid will occupy the bend of the tube, and the surface of contact will be in the branch occupied by the rarer liquid.

(5.) The phenomena of bodies denser than liquids, in which they are placed, sinking; and those which are rarer, rising and floating at the surface, are too familiar to need illustration.

(6.) The quantity of weight lost by a body immersed in a liquid, may be shown to be equal to the weight of an equal volume of the liquid, by means of a cylindric bucket which a solid of the same form exactly fills. If the bucket and solid be placed in one of the scales of a balance and counterpoised, and the solid be afterwards suspended from the scale, in a vessel containing a liquid, the former counterpoise will now preponderate; but if the bucket be filled with a portion of the liquid, equilibrium is restored.

(7.) The quantity of liquid displaced by a floating body, may be measured by performing the experiment in a graduated glass jar; and it will be found, that the increase in the space the liquid occupies, is exactly equal to a volume of the liquid whose weight is identical with that of the solid body.

(8.) The same experiment performed with two different liquids, and the same solid body, shows that the quantities the latter displaces of each respectively, are inversely as their densities.

(9.) Vases of different figures and areas, and therefore containing different bulks of a liquid, may be adapted by screws to one and the same base, held in its place by a counterpoising force; and it will be found that this opposing force is overcome in each of them, so soon as the liquid acquires the same depth. The same will be found true, whether the base be horizontal or inclined. The force that keeps the base in its place, may be a weight acting through the intervention of a lever of the first kind; this weight therefore furnishes a measure of the liquid's constant pressure in the vases of different figures; and this weight will be found, if the lever have equal arms, to be equal to the weight of a prism of water, having a section equal to the base on which the liquid presses, and an altitude equal to the depth of the liquid in the vessel.

(10.) An apparatus, called the hydrostatic bellows, is formed by connecting two circular or elliptical planks, by a flexible band of leather, in such a manner as to be water tight; to this a tall tube is adapted. If a liquid be poured into this apparatus through the tube, it will be found that it is capable of counterpoising a weight resting upon the upper plank, equal to the weight of a prismatic mass of water, whose section is equal to the area of the bellows, and altitude equal to that of the liquid in the tube.

(11.) The effect of pressures, exerted by means of pistons, may be best illustrated by the machine called Bramah's press, or by the forcing pump.

335. When a gravitating liquid is placed in a vessel, the pressure it exerts upon equal surfaces of its sides, will increase with the depth, as is obvious from the investigations of § 314; or as may be easily shown, by supposing the liquid to be divided into an infinite number of horizontal strata.

If the respective distances of these, from the surface, be z', z'', z''', &c., the respective pressures on equal surfaces, will be

$$sz'dk,\ sz''dk,\ sz'''dk,\ \&c.$$

quantities continually increasing with the variable co-ordinate, z. Hence:

Although the pressure upon a given base, horizontal, inclined, or vertical, depends upon the depth to which its centre of gravity is immersed below the level of the liquid, that point will not be, in the case of an inclined or vertical surface, the point of application of the resultant of the fluid pressures. This point of application is called the Centre of Pressure, and must be lower than the centre of gravity of the surface pressed. The true position of the centre of pressure, may be thus investigated in the case of a plane surface, in which it will always fall in the plane itself.

Let k represent the area of the surface, dk will be one of its small elements; let z represent the distance of the centre of gravity of the whole surface, from the upper level of the liquid, and z', z'', z''', &c., the distances of its several elements.

The several partial pressures will be

$$z\,dk,\ z'\,dk,\ z''\,dk,\ \&c.;$$

and the sum of their moments of rotation, §

$$dk(z^2+z'^2+z''^2+\&c.).$$

The whole pressure will be

$$Zk;$$

and if we call the distance of the centre of pressure, c, the moment of rotation will be

$$Zkc;$$

and this will be equal to the sum of the moments of the partial pressures, or

$$zkc = dk(.z^2 + z'^2 + z''^2 + \&c.) ;$$

whence

$$c = \frac{dk(z^2 + z'^2 + z''^2 + \&c.}{Zk} : \tag{360}$$

but we have

$$Zk = dk(z + z' + z''),$$

and, therefore,

$$c = \frac{dk(z^2 + z'^2 + z''^2 + \&c.}{dk(z + z' + z'')} ; \tag{361}$$

or if we call the several elements, A, B, C, &c., their respective distances from the surface, a, b, c, &c.,

$$c = \frac{Aa^2 + Bb^2 + Cc^2 + \&c.}{Aa + Bb + Cc + \&c.},$$

a formula identical with ( ), which we have found adapted to the investigation of the position of the centre of oscillation.

This may be given in a more convenient form.

Let x and y, be the vertical and horizontal co-ordinates of the surface, h the depth of its upper edge beneath the level of the liquid, the area will be

$$\int y\, dx ;$$

the several distances, z, z', z'', &c. ;

$$h + dx,\ h + 2dx,\ h + 3dx,\ \&c. ;$$

the expression (361) will, therefore, become

$$c = \frac{\int (h+x)^2 y\, dx}{\int (h+x) y\, dx} ; \tag{362}$$

and when the plane that is pressed by the liquid, reaches to the level surface of the latter,

$$c = \frac{\int x^2 y\, dx}{\int xy\, dx}. \tag{363}$$

From these formulæ we obtain the following useful results:

In a parallelogram, whose upper side is on a level with the surface of the liquid, the centre of pressure is at a distance of two thirds of its height beneath that surface.

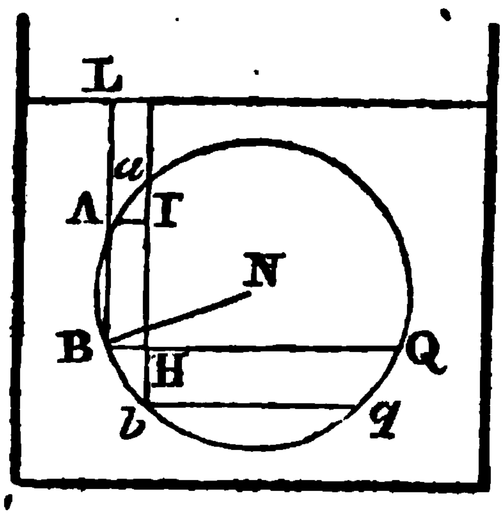
In a triangle, whose base is on a level with the same surface, this distance is one half its altitude; but if the vertex be in the surface of the liquid, the distance is three-fourths.

336. When a solid body is wholly immersed in a liquid, its surfaces undergo pressures; the measures of these will be the weight of columns of the liquid, having the surface pressed for a base, and the depth of its centre of gravity for the altitude. The whole amount of the pressures will, therefore, depend upon the magnitude of the surface of the solid, and have no reference to its

volume or density. But when the directions of these pressures are taken into account, they have a resultant which depends upon the volume solely, and is, as might indeed be inferred without investigation, the same with the loss of weight the solid experiences, or is equivalent to the weight of a mass of the fluid, equal in volume to the immersed solid.

A more strict investigation is, however, necessary:

Let the body immersed be, ABQ, and suppose its whole mass to be divided into an infinite number of small cylindrical columns, lying in a horizontal position. Let Bq be one of these, terminated at the surface in the bases, Bb, Qq, and distant from the level of the fluid by the depth $LB=z$.



Let BN be a normal to the surface, Bb, which may be considered as plane. Let bH be the projection of this surface upon a plane passing through b, and perpendicular to BQ. bH will be the section of the cylinder, perpendicular to its axis, and $bH=Bb$, cos. NBQ.

The liquid exercises upon the base, Bb, a pressure whose measure, taking the density of the liquid$=1$, is

$$z.Bb,$$

and whose direction is the normal BN. If this pressure be decomposed into three rectangular forces, that which acts horizontally in the direction BQ, will be

$$z.Bb \cos. NBQ=z.bH.$$

In the same manner we find the horizontal pressure on the opposite base, Qq, to be

$$z.qQ;$$

but the surfaces, bH and qQ, are equal to each other, and the horizontal pressures are, therefore, equal; they are also opposite to each other, and hence their resultant $=0$.

And in like manner it may be shown, that all the horizontal pressures parallel to BQ, mutually destroy each other.

If we next suppose the body to be divided into an infinite number of columns also horizontal, but at right angles to BQ, the same result will follow. And these two sets of forces at right angles to each other, make up the whole horizontal pressures; for each partial pressure may be resolved into two forces at right angles to each other, and all of these are included in the two sets of which we have spoken. The horizontal pressures upon the surface of a solid, immersed in a liquid, are, therefore, in equilibrio.

Next let us suppose the volume of the solid to be divided into an infinite number of vertical columns, and let one of these be Ab, terminated by the bases, Aa, Bb; the lower base being immersed to the depth, $LB=z$; the upper to the depth $LA=z'$.

The normal pressure upon the upper surface, Aa, resolved into three rectangular forces, gives for the vertical pressure,

$$z'.\mathrm{AI}.$$

The vertical pressure on the lower surface is in like manner,

$$z.\mathrm{BH},$$

and

$$\mathrm{AI}=\mathrm{BH}.$$

The resultant of these two opposite vertical pressures is therefore

$$(z-z').\mathrm{BH},$$

and is in the direction BA. It is, therefore, equal and opposite to the weight of the column of fluid that would occupy the space of the column Ab. And the resultant of all the vertical pressures would be equal and opposite to the weight of a mass of the liquid, equal in volume to the solid body immersed.

In the same manner it may be shown that the resultant of all the pressures, exerted by a liquid upon the vessel that contains it, is equal to the weight of the liquid itself.

This, however, does not contradict the law, that the amount of the pressures exerted outwards have for their measure the weight of a column of the liquid whose base is equal to the whole surface of the vessel, and whose altitude is the depth of its centre of gravity beneath the surface.

CHAPTER IV.

OF SPECIFIC GRAVITIES.

337. The specific gravity of a body is the ratio of its weight to the weight of an equal volume of some other body. In this general sense, it is equivalent to density, which is the relation between the weights of equal volumes of different bodies. But while density is an abstract term, and is determined by the direct comparison of the bodies in question, specific gravity is relative, and is a numerical value of the density obtained by comparison with some conventional standard. The standard, in general use for this purpose, is water. As this substance may contain gaseous, earthy, and saline impurities, it will only answer the purposes of a standard when freed from them. This may be effected by distillation. The heat of boiling drives off all the gaseous matter, and, in distillation, the solid substances are left behind in the still. Newly fallen rain water, at a distance from habitations, or that obtained by the melting of clean snow, is also sufficiently pure for the purpose.

338. Water, like all other substances, is liable to changes of volume by changes in its temperature; hence, it can only be employed as a standard, at some conventional temperature, to which the results of the experiments for determining specific gravities must be reduced. The English usually take, for this purpose, a mean atmospheric temperature, say about 60°; the French, the temperature at which water has its maximum of density. The latter is by far the most convenient and scientific method, for it is hardly possible to perform experiments at the exact temperature chosen by the English, in consequence of the practical difficulty of counteracting, by additions of colder or warmer water, the constant variations in the temperature of the air; and this method is dependent for its accuracy upon the indications of a thermometer, the divisions upon whose scale are arbitrary. On the other hand, it is always possible to obtain water at its maximum density, and easy to maintain it in that state. Thus a vessel of water, on which a small portion of ice floats, will have always in its lower parts, water of the maximum density; for so long as this state has not been attained by the water, the cooler portions will sink to the lower part of the vessel, in consequence of their greater weights under equal bulks; but so soon as the maximum of density is attained by the lowest portions of the liquid, this descent

ceases. Thus the French standard temperature is best suited to all cases where great accuracy is required, as it is better to obtain a result from direct experiments, that require no correction, than to correct those obtained under other circumstances.

In the cases that most frequently occur in practice, such nicety is unnecessary, and the experiment may be performed with water of any temperature; but the temperature must be noted, and a correction applied for it.

This correction rests upon the density of water, at the experimental temperature. We therefore subjoin a table of the densities of water, at different sensible heats, the maximum of density being employed as the unit.

TABLE.

Of the Densities of Water at different Temperatures.

32°.	0.9998918	50°.	0.9997825
34°.	0.9999428	55°.	0.9995324
36°.	0.9999761	60°.	0.9991886
38°.	0.9999944	65°.	0.9987549
39°.4	1.0000000	70°.	0.9982239
41°.	0.9999950	75°.	0.9976255
43°.	0.9999739	80°.	0.9969316
45°.	0.9999378	85°.	0.9961497

339. The densities of bodies might be compared, by immersing them, in succession, in a prismatic vessel, containing a liquid of less density than either. The liquid in which a solid is immersed, occupies a space as much greater than it did before, as is equal to the volume of the solid. Hence, in a vessel of constant and known area, the relation between the changes of level caused by the immersion of different solid bodies, gives the relation between their densities; and when the weight of a given bulk of water is known, the same method would give their respective specific gravities in terms of water as the unit.

This was the original method proposed by Archimedes, in the famous problem of the crown of Hiero.

The method now employed, depends upon the principle that a body, when immersed in a fluid, loses as much of its weight as is equal to the weight of an equal volume of the fluid. § 327.

Let S be the specific gravity of the solid body; s', that of the liquid; w, the absolute weight of the solid; w', its weight in the liquid; $w-w'$, will be its loss of weight in the liquid, and, therefore, the weight of a mass of the liquid, equal in volume to the

solid. If B be this common volume, the definition of specific gravity gives us

$$\left.\begin{array}{ll} & S=\dfrac{w}{B}\,,\ \text{and}\ s'=\dfrac{w-w'}{B}\,; \\ \text{whence} & \\ & B=\dfrac{w}{S}\,,\quad B=\dfrac{w-w'}{s'}\,, \\ & \dfrac{w}{S}=\dfrac{w-w'}{s'}\,, \\ \text{and} & \\ & S=s'.\,\dfrac{w}{w-w'}\,. \end{array}\right\}\qquad (364)$$

If the liquid employed be pure water, at its standard temperature,

$$s'=1\,;$$

and

$$S=\frac{w}{w-w'}\,. \qquad (365)$$

To obtain then the specific gravity of a solid body, its weight must be divided by the loss of weight in pure water, of the standard temperature. If the water be at any other temperature, we obtain the approximate specific gravity, by dividing the weight of the body by its loss of weight in the water; this may be reduced to the true specific gravity, as will be seen from the final formula, (364), by multiplying the proximate specific gravity by the density of water, at the temperature at which the experiment is made; the density of water at the standard temperature being taken as the unit. For this purpose, the table in the preceding section may be used.

340. In determining specific gravities, we use a common balance, to which apparatus intended to facilitate the process of weighing the body in water, is adapted. When the solids are in the form of masses united by the attraction of aggregation of their particles, and are denser than water, the process is extremely simple. The body is weighed in air, and then in water, and the first weight is to be divided by the difference of the two weights; a correction is then applied, as has been just stated, for the temperature of the water, if it be not at its maximum density.

When the solid is in a state of fine powder, it must be placed in a vessel whose weight in air and in water, have been previously determined. When the vessel containing the powder is weighed in air and in water, the difference between these, and the weights of the vessel alone, in air and in water, are the weights of the powder under the two different circumstances. The calculation may then be performed as before.

When the solid is soluble in water, it often happens that a liquid may be found in which it is not soluble. The density of this liquid being known, by methods hereafter to be explained, the body may be weighed first in air and then in this liquid; we obtain, by the same method of calculation, its specific gravity, in terms of the liquid as the unit. This, as will be seen from (364) must be multiplied by the specific gravity of the liquid, in order to obtain that of the solid in terms of water; for it is obviously unimportant whether we use water at a temperature different from the standard, or a liquid whose density in respect to water is known.

When the solid is less dense than water, its weight in that liquid cannot be obtained directly; but it may be attached to another, sufficiently dense to cause both to sink. The weight of the light body in water will obviously be the difference in the weights of the two bodies when united, and of the heavy body alone.

The principle and method of calculation may be best illustrated by symbols:

Let l be the weight of the light body; h, that of the heavy body, air; h', of the same body in water; c, the joint weight of the two bodies in air; c', their joint weight in water. Then $c-c'$, and $h-h'$, will be the respective losses of weight, and the loss of weight of the light body will be

$$(c-c')-(h-h');$$

hence,

$$S=\frac{l}{(c-c')-(h-h')}. \qquad (366)$$

The divisor, in this formula, will always exceed the dividend, and the resulting specific gravity is a fraction, which is usually obtained in the decimal form.

When no convenient fluid can be found in which the solid is insoluble, its specific gravity cannot be obtained by means of the hydrostatic balance.

341. The specific gravity of a liquid may be determined with the hydrostatic balance, in various ways.

(1) A solid of convenient size and form, usually a bulb of glass of known weight, may be weighed in water, and in the liquid whose specific gravity is sought. The respective losses of weight, are the weights of equal bulks of the two fluids; and water being the unit, we divide the loss of weight in the liquid by the loss of weight in water.

For if we call the loss of weight in water, w''; the loss in the other liquid, l, the formula (365) becomes

$$S=\frac{l}{w''}; \qquad (367)$$

(2.) A phial of known weight may be taken, filled with water and weighed; the increase of weight is the weight of the contained water; it is then emptied, and filled with the liquid, whose weight is obtained in the same manner: we have again weights of equal bulks of the substances, whence the specific gravity may be obtained, as in the preceding instance.

(3.) We know, by accurate experiments, the weight of a given bulk of water; hence, if a phial of known weight and internal capacity be taken, the weight of the water it contains is known. It is, therefore, only necessary to fill the phial with the liquid whose specific gravity is sought, and weigh it. The mode of calculation is obvious.

342. In determining the weight of a given volume of water, or of any other liquid, it is found much more easy in practice to weigh a solid of known dimensions and weight in the liquid, than to make a vessel of a given capacity, and fill it with the liquid. This arises from the great ease and certainty with which the external dimensions of a solid can be measured, compared with those with which the internal capacity of a vessel can be guaged.

Sir George Shuckburgh, in order to ascertain the weight of a given bulk of water, (a cubic inch,) in Troy grains, made use of three solids of the same material, but of different forms. One was a sphere, another a cylinder, the third a cube: the material was brass. These having been constructed with every possible care, were afterwards measured by a scale furnished with powerful microscopes. In this way the lineal dimensions were ascertained, and the inequalities, inseparable from the best materials, and most accurate workmanship, detected: from these data, their volumes were computed.

Experiments, with the same three solids, were also made by Kater, in the researches on which the British standard of weights and measures is founded.

In the French investigations for establishing the basis of their metrical system, a similar method was used, with a body of cylindrical form, and of larger dimensions than those used by Shuckburgh.

The experiments of Kater make the weight of a cubic inch of pure water at the temperature of 62°, 252 grs. 458, Troy.

The State of New-York has, as stated in Book IV., Chapter V., taken for the basis of its standard of weight the avoirdupois pound, of such magnitude that the cubic foot of pure water, at its maximum density, weighs 1000 oz. or 62½lbs.

The same difficulty that we have stated, exists in determining standards of cubic measure, by means of the internal capacity;

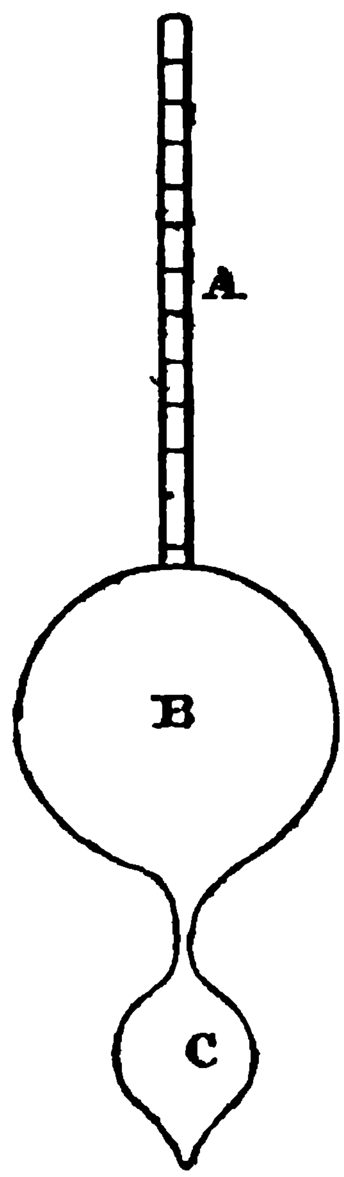
hence, the British Government, and the Legislature of the State of New-York, have defined their units of capacity, by prescribing the number of pounds of distilled water, of the standard temperature that the vessel shall contain. This method is preferable to that of defining it by the number of cubes of the measure of length that it shall comprise.

343. In the instructions for determining specific gravities of § 338, it will have been noted that the weight in air is spoken of, instead of the absolute weight referred to in the theory of § 337. These are not identical, for the air being a fluid, presses upon the solid, and produces a loss of weight analogous to that caused by liquids, as demonstrated in § 325. Indeed, as the air has an uniform density within the space the solid under experiment occupies, the effect is in fact identical, and the solid loses as much of its absolute weight as is equal to the weight of its volume of air. In different bodies, of equal weight, this loss will be in the inverse ratio of their densities, and thus the rarer bodies will be the most affected. In very accurate investigations, it becomes necessary to take the buoyancy of the air into account. The principle on which this correction is founded, may be inferred from the theory that has already been laid down. The exact amount, and the variations to which it is subject, must be deferred until we have investigated the properties of air.

When the weights and the substance to be weighed are homogeneous, both are equally affected by the buoyancy of the air. Hence, there are a few cases in which no correction is necessary, even in the most accurate investigations: thus, in prescribing the weight of a given measure of distilled water, for the purpose of defining the unit of weight, the British and New-York statutes declare that the experiments shall be made with brass weights, under which denomination the experimental solid is included.

344. In many practical cases, methods more speedy than are furnished by the hydrostatic balance, are necessary. This necessity occurs more frequently in the determination of the specific gravities of liquids, and particularly in ascertaining the value of spirituous liquors, both for the convenience of commerce, and the collection of revenue. In these cases, an instrument called the Hydrometer, is employed.

A hydrometer consists essentially of three parts: a stem, on which divisions are drawn; a bulb or hollow float, to render it buoyant in a liquid; and a weight by which the stem may be made to float in a vertical position. These parts are exhibited in the following figure, in which A is the stem, B the bulb, and C the weight.



The principle on which the hydrometer is used, is, that a body set to float on two different fluids, of less density than itself, will displace volumes of them, that are inversely as their respective densities. § 330.

It is, therefore, obvious, that the hydrometer must be less dense than the rarest liquid in which it is intended to be used; and that in the most dense, it must sink at least so far as to cover its bulb; otherwise, it would in the one case sink to the bottom, and in the other, the liquid would not reach the divisions of the stem.

Delicacy of indication will be promoted in the hydrometer by making the stem slender, in which case the differences of the depths to which it sinks will be considerable for small changes of specific gravity. On the other hand, the extent of differing specific gravities, or what is styled the *scale*, will be limited, when the stem is slender, unless it be made of inconvenient length. When great accuracy is required, the stem must be necessarily slender; and in order to obtain measures of specific gravities, not included within the scale of a single instrument, two or more hydrometers must be used.

This is generally the case with the hydrometers used by chemists, of which the material is glass. Glass is well adapted to the purpose, from its cleanliness; the case with it may be fashioned by the blowpipe; and from its being acted upon by but few chemical liquids. On the other hand, its fragility is an objection to its use by manufacturers, and in the hands of fiscal officers.

The necessity of having more than a single instrument, may be obviated by adapting moveable weights to the same hydrometer, in such manner that they may be removed, or added, according to the density of the liquid.

For these reasons the hydrometers in more frequent use, are made of metallic substances, and have metallic weights.

Dicas' hydrometer, which is prescribed by law in the customs of the United States, has 36 moveable weights, and the stem has 10 divisions: the weights are so adjusted that in a given liquid, if the hydrometer sink with one of the weights to the 0 on the stem, the next weight shall sink it 10°. Hence the divisions amount to 360. This instrument is extremely delicate, and well

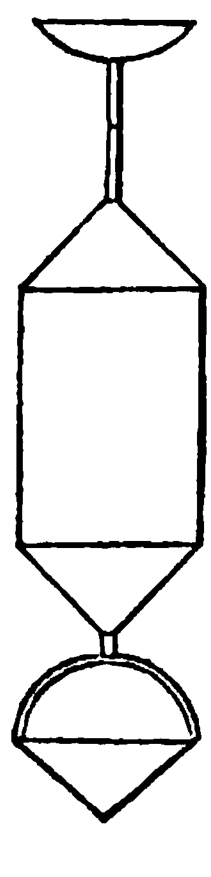
suited for nice investigations, but is too complex and troublesome in its use for ordinary purposes.

A hydrometer upon similar principles, but of greater simplicity, has also been constructed in Boston.

A still more simple instrument has been introduced by Southworth, of New-York. The material is also silver, and there is but one moveable weight. All these instruments measure specific gravities from that of pure water, or 1, to that of alcohol or 0.825.

Silver is objectionable as a material for hydrometers, unless it be gilt, in consequence of its liability to tarnish, and the necessity of cleaning it, which will wear away the metal, and alter the indications of the instrument.

345. The hydrometer may be used for determining the weights and specific gravities of solids. For this purpose the instrument is made so much lighter than water, as to require the addition of a considerable weight to sink it to a mark on the middle of its stem. This weight being known, in some conventional unit or standard, the body to be weighed is placed upon a cup prepared for the purpose at the top of the instrument; weights are added until the hydrometer sink to the mark at which it before stood; the joint weight of the body, and the additional weights, is, therefore, the same as that which was found by experiment to bring the instrument to this position. It only remains to subtract the weight added when the body is placed in the cup, from that which when used alone sinks it to the proper level; the difference is the weight required.



If the specific gravity is to be ascertained, the body is next placed in a second cup, adapted for the purpose, to the top of the weight that steadies the instrument, where it is of course immersed in the water. Its loss of weight will be apparent by a rise in the stem of the instrument; fresh weights are added to bring it again down to the original mark, and these are the measure of this loss.

Such is the hydrometer of Nicholson, the form and arrangement of which may be understood from the inspection of the annexed figure.

An instrument for weighing, founded upon similar principles, but far superior in accuracy, has recently been constructed by Mr. Hassler, to be employed in the rectification of the weights and measures used in the customs of the United States. A hol-

low bulb of glass, of the figure of a flask, is placed in a vase that is set upon a bracket; three steel rods are sealed to the neck of the flask, and bear a metallic plate, cut in such a manner as to form three horizontal arms that project beyond the sides of the vase; from these arms three rods proceed downwards, two of which embrace the bracket, and the third descends in front of it; these rods bear a shelf or scale on which weights and the substance to be weighed are placed. By the use of the bracket, the position of the weight is thus brought beneath the vase, and the equilibrium of the apparatus is rendered stable, while in Nicholson's hydrometer the equilibrium is tottering. The operation of weighing may, therefore, be performed with greater ease. The size of the instrument is also much greater than is ever given to Nicholson's; and as the whole exposed to the liquid is of glass, except the rods, mercury may be used instead of water, by which liquid the capacity of the instrument to bear heavy weights, is increased more than thirteen-fold.

346. The specific gravity of bodies, is one of their most important distinctive characters, and must hence be included among the properties by which they are described and defined. By means of it, we may frequently determine the nature of bodies, without having recourse to any other means; and in all cases it is an aid in the discovery of the class to which substances hitherto unexamined are to be referred. We therefore use the method of specific gravities in many branches of physical science, in several of the arts; and although in commerce it is as yet applied to but few purposes, its use is capable of much farther extension.

The seven ancient metals, which are still in most frequent use, have such marked differences in density, that they may frequently be distinguished from one another by their specific gravities; and if alloyed or adulterated, the determination of the specific gravities of the compounds will, in many cases, detect the mixture.

Thus in gold, whether in the shape of bullion, or of coin, great specific gravity is often a test of its value. Until recently it was a sure one; for gold, before the discovery of platinum, was the densest of all known substances, and any admixture is sure to produce a diminution in the specific gravity. Recently an alloy of platinum has been discovered, that possesses many of the external characteristics of gold. Platinum being the densest of substances, we now know that even the appropriate density of alloys of gold may be given by it, to the alloys of which the former metal forms a part. These, however, want the ductility and malleability of gold, and when they have the same colour, have less density than its alloys. Among the other ancient metals, tin is the least dense, and hence when impure, its specific gravity is increased.

The adulterations of drugs, and pharmaceutical preparations, may be frequently detected by the change of specific gravity they produce. Acids, in particular, can have their purity tested in almost all cases.

In the operations of practical chemistry, the strength of solutions, the proper state of concentration suited for crystallization, for fermentation, and distillation, are known by the use of the hydrometer. This instrument is therefore of value in the arts of brewing, distilling, sugar refining, as well as in the processes that are strictly chemical.

The value and character of metallic ores, may also be in many cases inferred from their specific gravity; and in general it forms a marked distinction of minerals, by means of which they may be classed, and their constituent parts inferred.

Such are a few of the more important purposes to which the method of specific gravities, is applicable.

347. When two substances, whose specific gravities are known, are mixed mechanically, it might at first sight be inferred that the proportions in which they exist in the compound, would be readily inferred from its specific gravity. This will be apparent from the following investigation:

Let w and w' be the weights of the two bodies;
g and g' their respective specific gravities;
G the specific gravity of the compound.

The weight of the compound is $w+w'$. The weight of its volume of water (364) is

$$\frac{w+w'}{G}.$$

The respective weights in water, of the volumes of the two components, are

$$\frac{w}{g} \text{ and } \frac{w'}{g'};$$

hence

$$\frac{w+w'}{G}=\frac{w}{g}+\frac{w'}{g'}; \tag{368}$$

whence we obtain

$$w=\frac{(G-g')g}{(g-g')G}(w+w'); \tag{369}$$

and

$$w'=\frac{(g-G)g'}{(g-g')G}(w+w'). \tag{370}$$

These formulæ are, however, of little or no value in practice;

for when two bodies are mixed, their joint volume is rarely or never the same as the sum of their respective volumes. In general, the joint volume is less than the sum of the separate volumes. This diminution of volume, that occurs in most cases of mixture, is called Concentration. The most remarkable instance of concentration that has been noted, is that stated by professor Robinson, who says, that when 40 pts. of platinum are alloyed with 5 pts. of iron, the bulk of the mass is but 39. This alloy, then, of the densest simple substance with one of little more than a third of its density, gives a compound even more dense than the first. In the case of the mixtures of alcohol and water, that form the spirituous liquors of commerce, the concentration produces marked effects, and must be taken into account in all the estimates of their value that are drawn from their specific gravities. The densities of such mixtures have been made the subject of accurate experiments by Gilpin, the results of a part of which are comprised in the following

TABLE

Of the densities of mixtures of alcohol and water, at the temperature of 60° of Fahrenheit.

Pts. of Water.	Pts. of Alcohol.	Specific Gravities.
10	0	1.00000
10	1	0.98654
10	2	0.97771
10	3	0.97074
10	4	0.96437
10	5	0.95804
10	6	0.95181
10	7	0.94579
10	8	0.94018
10	9	0.93493
10	10	0.93002
9	10	0.92449
8	10	0.91933
7	10	0.91287
6	10	0.90549
5	10	0.89707
4	10	0.88720
3	10	0.87569
2	10	0.88208
1	10	0.84568
0	10	0.82500

We have not reduced this table to the standard of the maximum density of water; for the habitual custom, not only in England, but in France, where the spirits of commerce are concerned, is to reduce the densities at the observed temperatures, to that of 60°. Upon this principle the hydrometer or alcoometer, planned by Gay Lussac, and used in the French excise, is graduated.

In ascertaining the values of spirituous liquors, for the purpose of the collection of duties in the United States, four different stages of proof are established by law; each of these pays a specific duty. The spirit that lies between any two of these in density, is counted as belonging to the lower proof. This method is unfair in practice, and has been made the source of actual, if not of technical frauds upon the revenue. The true and equitable principle is that adopted by the French government; by this the duty is estimated upon the quantity of alcohol of the specific gravity of 0.825, at the temperature of 60°, that is contained in the mixture.

348. One other method of determining the specific gravities of liquors remains to be mentioned. A number of bubbles may be blown from a glass tube, having different densities. Their specific gravities, having been determined by experiment, in liquids of known densities, are marked upon them.

When the specific gravity of a liquid is to be determined, they are thrown into it, until one is found that remains quiescent in any part of the liquid in which it is placed, and with the specific gravity of this bubble, that of the liquid is identical.

349. We subjoin a table of specific gravities compiled from the most accurate authorities.

TABLE

Of the specific gravities of bodies, in terms of water, at its maximum of density.

Platinum,	Rolled,	22.6667
	Wire,	21.0396
	Hammered,	20.3356
	Purified,	19.4981
Pure gold.	Hammered,	19.3600
	Cast,	19.2562
Gold 22 carats fine cast,		17.4846
do 20, do " cast,		15.7075
Tungsten,		17.6000
Mercury,		13.5967
Lead,		11.3512
Palladium.		11.3000
Rhodium,		11.0000

Silver cast,		10.4743
Bismuth,		9.8220
Copper,		8.7880
Molybdenum,		8.6110
Arsenic,		8.3080
Nickel,		8.2790
Uranium,		8.1000
Soft Steel,	Hammered,	7.8397
	Cast,	7.8324
Hard Steel,	Hammered,	7.8173
	Cast,	7.8156
Cobalt,		7.8119
Bar Iron,		7.7873
Tin,		7.2907
Cast Iron,		7.2083
Zinc,		6.8604
Antimony,		6.7120
Tellurium,		6.1150
Chromium,		5.9000
Iodine,		4.9480
Sulphate of Baryta,		4.4300
Zircon,		4.4157
Ruby and Sapphire, (oriental)		4.2830
Diamond, from		3.5307
to		3.5007
Flint Glass,		3.3290
Fluor Spar,		3.1908
Tourmaline,		3.1552
Native Sulphur,		2.0329
Sodium,		0.9726
Potassium,		0.8651
Sulphuric Acid,		1.8408
Nitric Acid,		1.5500
Water from the Dead Sea,		1.2402
Nitric Acid,		1.2176
Sea Water,		1.2062
Milk,		1.0300
Distilled Water,		1.0000
Bourdeaux Wine,		0.9938
Burgundy Wine,		0.9214
Olive Oil,		0.9152
Muriatic Ether,		0.8740
Spirits of Turpentine,		0.8697
Naptha,		0.8475
Standard Alcohol of Gilpin,		0.8250
Alcohol perfectly pure,		0.7920
Sulphuric Ether,		0.7155

350. If the weight of a given bulk of water be known, the Hydrostatic balance may be applied when geometric methods fail, to determine the volume or cubic contents of bodies. For this purpose they must be weighed in air, and in water, and as their loss of weight is the weight of their volume of water, their cubic contents may be calculated.

Their volume B in cubic inches, when the weights are taken in troy grs., will be

$$B=\frac{w-w'}{252.458}. \tag{371}$$

Their volume in cubic feet, when the weights are taken in avoirdupois ounces will be

$$B=\frac{w-w'}{1000}. \tag{372}$$

When the specific gravity of a body is known, we may calculate its volume when its weight is given, or its weight when its volume is given upon similar principles.

If B be the volume, and W the weight of the body, g its specific gravity, w the weight of the cubic unit of water in the unit of measure, we have

$$W=Bgw, \tag{373}$$

and

$$B=\frac{W}{gw}. \tag{374}$$

CHAPTER V.

Of the Nature and Characters of Elastic Fluids, and of the Pressure of the Atmosphere.

351. We become acquainted with the material existence of the air that composes the atmosphere of our earth, and which is not at first as obvious to our senses as that of solids and liquids, in various ways:

(1.) We observe it moving in currents of wind, capable of producing powerful mechanical effects;

(2.) We find it resisting the entrance of other substances into the space it occupies, thus: when a glass jar is inverted and pressed into a mass of water, the water enters it at first to but a small distance; and whatever be the depth to which it is forced down, the air will still occupy a part of the jar;

(3.) We may weigh it, and thus show that, like other material substances, it is affected by the attraction of gravitation.

The fluid nature of air may be shown by the freedom with which bodies at the surface of the earth move in it; and by its exerting pressures in all directions, and therefore upwards as well as downwards: thus, if a glass vessel be filled with water, and a piece of paper be laid upon the surface of the water, projecting over the edges of the vessel; the vessel may be carefully inverted without spilling the water; and the liquid will afterwards remain in the vessel, in which it is supported by the pressure of the air.

The experiment (2) may be extended to show another essential property of air, namely, its elasticity; for, as the vessel descends in the mass of water, the water rises, although, as has been stated, it never wholly fills it. The space occupied by the air, is therefore lessened by the pressure of the liquid; if the vessel be permitted to rise, the water descends, until the mouth of the vessel reach the surface, when the air again occupies the same space it did at first. It is, therefore, capable of having its bulk diminished by pressure, and of restoring itself to its original volume when the pressure is removed, or is elastic.

352. This elasticity is considered as permanent, for, no mere change of temperature, or in its relations to latent heat, will convert atmospheric air into a liquid or solid form. It is indeed said that intense pressure will reduce it to a liquid state, and we know that its chemical constituents do, when in combination, become

both liquid and solid. Still the difference in this respect, is so marked between the air of our atmosphere and another class of elastic fluids, that we are not warranted, in treating of its mechanics, to refuse to receive the permanence of its elasticity, as one of its principal characteristic properties.

The researches of chemistry have shown us that the air of our atmosphere is not homogeneous in its chemical constitution, but that it is a mixture of several fluids of the same mechanical nature, differing in their chemical properties. So, also, have a number of other fluids permanently elastic in character, that form no perceptible portion of the atmosphere, been discovered; to the whole of these we give the common epithet of Gas.

A gas, then, is a material substance, belonging to the class of fluids, gravitating or possessed of weight, and permanently elastic.

353. When water is heated in a close vessel, an elastic fluid is generated; this finally acquires such expansive force as to break the vessel: if boiled in a glass vessel, with an open neck, the space, not occupied by water, will appear perfectly transparent and colourless; but a cloud will appear to issue from the neck. This cloud, if examined, will be found to be composed of water in a state of minute division, and we infer that the space within the vessel is filled with an elastic fluid. To render this more evident, adapt a syringe to the neck of the vessel, containing a piston fitted air tight, and depressed to the bottom of the syringe; when the water begins to boil, the piston will begin to rise, and will speedily reach the top of the syringe; if cold water be now applied to the exterior of the syringe, the piston will be caused suddenly to descend; we infer, therefore, that the elastic fluid, that had been generated is now condensed. The visible cloud that issues from a vessel, was formerly called vapour or steam; we now apply those terms to the invisible elastic fluid; and we distinguish it from gases, as being condensible by physicial means, which they are not.

Numerous experiments show, that water is not the only substance, that is, when heated, converted into an elastic fluid; few or no substances indeed exist, that do not become volatile at a greater or less degree of temperature, and all these volatilized matters are condensed when the heat is withdrawn. The name of vapour is hence applied to them all.

Vapour, then, is a material substance, existing in the fluid form, elastic, but not permanently so, being capable of condensation by cold.

We therefore subdivide elastic fluids into two classes: Gases, or permanently elastic fluids; and Vapours, or condensible elas-

tic fluids. Physically speaking, the line that separates these two classes, is not distinctly marked. There are some substances classed as gases, that are condensible with no great difficulty; thus ammonia becomes liquid at very low temperatures; indeed, few or none of the gases are capable of retaining their elastic form under intense pressure; for even atmospheric air has, as stated by Perkins, been reduced to the liquid form. On the other hand, there are some vapours that, under physical or mechanical circumstances, differing in no great degree from those we find existing at the surface of the earth, would appear permanently elastic: thus, the vapour of ether is formed under ordinary circumstances at 98°, and would, in a climate of that temperature, be gaseous.

The elastic force with which gases or vapours tend to expand themselves, is called their Tension.

354. The tension of elastic fluids is increased by heat, and diminished by cold; thus, under a given pressure, a given weight of atmospheric air, and in general, of any elastic fluid, is found, when heated, to occupy a greater space. This may be simply shown by enclosing a portion of air in a bladder, and exposing it to heat; the bladder will be distended and finally burst. If the bladder be completely filled, and then cooled, it will cease to be distended, and the distention may be restored by raising it again to its original temperature.

355. The mass that composes the atmosphere of our earth being a fluid, the general properties of that class of bodies would lead us to infer, that it will tend, from the freedom of motion among its particles, to distribute itself uniformly over the whole surface of the earth. This is confirmed by experience, for we find the atmosphere exerting a pressure, that taken at a mean, is constant at every point upon the mean surface of the earth.

This property is not confined to the mass, but is inherent in each of its several chemical constituents. We learn this from the researches of Dalton, who has shown conclusively that every different elastic fluid tends to distribute itself uniformly over the surface of the earth, and thus to form a separate atmosphere of itself. This tendency is exerted precisely as if no other elastic fluid were present; but the rapidity with which the distribution takes place, is affected by the fluid resistance of the other elastic fluids: and we shall see, hereafter, that in the present physical state of our planet, the several elastic fluids that make up our atmosphere, are constantly tending to a state of equilibrium, that they never completely attain. This discovery of Dalton, we shall find of great value; for the present, we shall merely state, that it has fully explained the fact, observed by chemists, but not

to be accounted for upon the principles of that science alone, of the constant proportion of the two gases that compose the principal part of atmospheric air.

356. The fluid pressure of the atmosphere was first exhibited by Torricelli. It having been found that the height to which water rises in a common pump, was limited; and it being inferred by him that the cause was mechanical, he saw that this, whatever were its nature, must on the principle of § 326, raise a mass of denser fluid to a less height. He, therefore, inferred that the experiment of the pump might be performed with an apparatus of less size, provided he used mercury instead of water. Taking, therefore, a glass tube of about three feet in length, he adapted a piston to it, and plunged the end in a basin of mercury. On drawing up the piston, the mercury followed until it reached the height of 30 inches, at which the fluid ceased to rise; on drawing up the piston farther, a space was left between it and the surface of the mercury. Now as the limit to which water rises, in a well-constructed common pump, is 34 feet; and as this height is to 30 inches in the inverse ratio of their respective densities, (See Table of Specific Gravities, § 346), the pressures of these two columns of the liquids are equal, and require an equal force for their support, if the force be not identical. Torricelli at once, and as was considered at the time, with great boldness, inferred that the forces which supported the water in the pump, and the mercury in the tube, were identical, and that it was to be sought in the fluid pressure of the atmosphere. As the weight of the atmosphere was then unknown, and the fact of its compressibility not understood, the boldness of his inference staggered even some of the most enlightened of his cotemporaries.

Toricelli also planned a more convenient mode of performing his experiment. Taking a tube of the same length as before, he closed it at one end, and filled it with mercury; placing the finger upon the open end of the tube, he inverted it in a basin of the same liquid; on removing the finger, the mercury subsided from the sealed end of the tube, until it reached the height of about 30 inches above the level of the mercury in the basin, at which it remained stationary.

That the cause which supports a column of water, is identical with that which sustains the column of mercury, in these experiments, may be shown, by pouring water upon the surface of the mercury in the basin. If the tube be now raised until its open end be above the level of the mercury, but remain still immersed in the water, the mercury in the tube will, from its superior weight, fall through the water beneath it; the water will

enter the tube to supply its place, but so far from ceasing to rise at a height of 30 inches, rushes to the top of the tube, and would fill it, even were its length as great as 34 feet.

Pascal, who did not at first admit the conclusions of Torricelli, planned an experiment which furnished complete evidence of the accuracy of the views of that philosopher. It has been shown, § 316, that that the pressure of a fluid, upon a given base, is a function of the depth. If then the cause of the rise of the mercury be the pressure of the atmosphere, it is obvious that on carrying the Torricellian apparatus to a greater height above the level of the sea, the pressure must diminish, and the column of mercury that it will support will be lessened in height. This experiment being performed at the base, and on the summit of the mountain *Puy de Dome*, in Auvergne in France, gave this anticipated result; a diminution in the height of the column of mercury being found amounting to about 3 inches, for a change of level of 3600 feet.

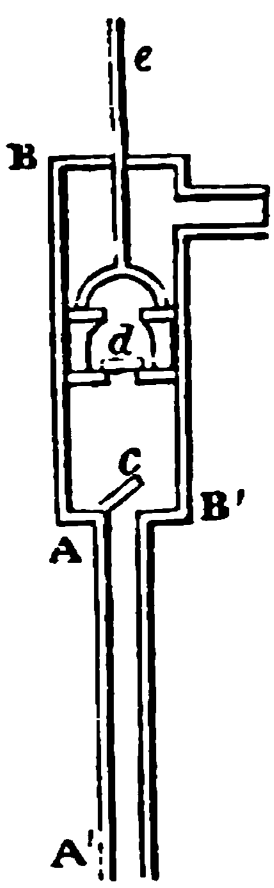
357. The apparatus of Torricelli, then, demonstrates conclusively the existence of atmospheric pressure. It also enables us to measure its amount. For the pressure of the column of mercury, with which that of the atmosphere must be in equilibrio, is, § 331, equal to the weight of a prism of the fluid whose height is 30 inches. Upon a square inch, the bulk of this prism is 30 cubic inches; and this quantity of mercury has a weight that differs but little from 15 lbs. The air of the atmosphere, therefore, presses, at the level of the sea, with a force equivalent to 15 lbs. upon each square inch of surface. This quantity of 15 lbs. per square inch, constitutes that unit in which the pressures and tensions of elastic fluids are measured, and which is called an Atmosphere.

Were atmospheric air a fluid of uniform density throughout, its height above the mean surface of the earth might be calculated from the Torricellian experiment; for if we call the density of water 1000, that of mercury is 13600; and that of air at a mean, at the level of the sea is usually estimated at $1\frac{3}{5}$; the mean height of the column of the mercury in the barometer, at the same level, is thirty inches, or $2\frac{1}{2}$ feet. Calculated from these data, the height of an atmosphere of uniform density is 27600 feet.

The Torricellian apparatus, with additions and under modifications that will be hereafter described, goes at present by the name of the Barometer.

358. The common pump, to an observation on the action of which the Torricellian experiment was due, is an apparatus whose

form and mode of action may be understood from the annexed figure. AA′, is a pipe of any convenient length, less than the limit to which water may be raised by atmospheric pressure; BB′, a barrel generally of greater diameter than the pipe; *c*, a valve opening upwards, at the junction of the pipe and barrel; *d*, a piston, moveable by means of the rod *e*; in this piston there is also a valve opening upwards. When the piston is raised, the air in the barrel between the two valves is expanded, and its tension diminished; the air in the barrel, therefore, opens the valve *c*, and the whole mass of air tends to become less dense; but as this pipe communicates with water in a basin, that is pressed at its surface by the air of the atmosphere, this pressure causes the water to rise in the tube, until the tension of this confined air becomes equal to the pressure of the atmosphere. On depressing the piston *d*, the valve in it opens, and air passes upwards from the barrel as the former descends; but the valve *c* is closed by the downward pressure, and the volume of water that has entered the pipe, remains. On again raising the piston, the same action takes place as at first, and an additional quantity of water enters the pipe: a second depression of the piston causes the same effects as the first. Thus a column of water will be raised by the alternating motion of the piston, until that fluid reaches the piston in its lower position. On raising the piston when the water has reached it, that fluid will be compelled to follow it by the pressure of the atmosphere; when the piston is again depressed, the water flows through the valve situated in it; and on its being again raised, the water will be lifted upon its surface until it reaches and flows out of the spout F.



Although in theory the limit of the height to which water may be raised by the common pump, measured from the surface in the basin beneath, to the highest position of the moveable piston, is 34 feet; it is not found practicable, with pumps of ordinary structure, to raise that liquid more than about 28 feet. This difference arises from the difficulty of making the apparatus absolutely air-tight.

Liquids are supported in a syphon, upon the same principle that they are raised in the common pump. If a bent tube be filled with a liquid, and inverted, without permitting the liquid to escape, and one of its branches be immersed in a vessel containing a mass of the same liquid, so much of the liquid in the tube as is above the level plane, of which the surface of that in the vessel

forms a portion, will be supported by the pressure of the atmosphere; and the columns, in the two branches of the tube, lying above this level, exactly balance each other; but the remainder of the column, in the branch that is not immersed, will cause the column of which it is a part, to preponderate and descend; the pressure of the atmosphere will cause an equal quantity of liquid to rise in the opposite branch to supply its place, and thus a continual stream will flow from the open end of the tube, until the level of the liquid in the vessel descends as low as the orifice on the branch of the tube that is not immersed.

The apparatus called the Syphon, is a bent tube, and has usually branches of unequal length, the shorter of which is immersed in the vessel. Hence the flow of liquid will continue until its level descends as low as the opening in the immersed branch; the air will then enter the syphon, and the whole of the liquid will be discharged from it.

Instead of filling the syphon, and inverting it, the air may be extracted by a pump, or by the action of the lungs; the mouth is, in the latter case, applied to a pipe adapted for the purpose, and the end of the syphon that is not immersed, closed by the finger, or by a stopcock. The apparatus, in this form, is called a Crane, and is used for the purpose of decanting liquids. The flow of liquid from a syphon, will have a retarded velocity, provided the vessel be permitted to empty itself. But if it be kept full, by any appropriate means, the velocity remains constant.

CHAPTER VI.

Of the Air Pump.

359. The mechanical properties of the air may be best investigated experimentally, by means of the apparatus called the Air Pump.

If an instrument similar in construction to a common pump, but of more accurate workmanship, be adapted to a tight vessel, the action, which in the common pump diminishes the tension of the air in such a manner as to allow water to be forced up by the presure of the atmosphere, will now act merely to rarefy the air contained in the vessel. When the piston is raised, the air in the vessel expands in consequence of its elasticity, opens the lower valve, and fills the whole space beneath the piston with air of an uniform tension, but of diminished density. When the piston is depressed, the air contained between it, and the lower valve, is compressed until it reach the same density as the external atmosphere ; it then opens the upper valve and passes out. On raising the piston again, a farther rarefaction takes place ; and thus, at each alternate motion of the piston a new expansion takes place, and a portion of the air originally contained in the vessel, passes out.

We call the space within the receiver, from which the air has been extracted, the Vacuum of the Air Pump. This is less perfect than that at the top of the barometer, which is called the Torricellian Vacuum.

The first air pump was invented by Otto Guerické, of the city of Magdeburg, and had the form we have used for our illustration. The joint between the pump and the close vessel, was rendered air tight by a collar of leather, moistened with water. We now use in the several joints of the pump, leather dipped in a liquid oleaginous matter.

The air pump has received many modifications in form, and improvements in structure, since the time of its invention, of which the following are among the most important.

A close vessel is ill suited for the purpose of introducing substances, or apparatus for experiment, into the vacuum of the air pump ; but if an open receiver be taken, and placed upon a plate, with an orifice in the middle of which the pump communicates by a pipe ; when the pump is set in action, it will tend to rarefy the air in the receiver, and the pressure of the atmosphere will fix the latter to the plate of the pump. The receiver will then,

if the joints between it and the plate be air tight, be to all intents and puposes a close vessel. To render this joint air tight, the plate was at first covered with a piece of oiled leather. It was however found, that the evaporation of the oil from the leather, lessened the extent of rarefaction, by continually supplying a fresh elastic fluid. For this reason, then, the plate of the pump (usually made of brass) has been ground; the rim of the receiver is also ground, in order to adapt itself to the plate. In this way the joint will frequently be air tight, without any other precaution, but more generally it is necessary to touch the rim of the receiver with a little oil.

As oil is acted upon by brass, and corrodes it, glass plates have been substituted for brass, and are ground in the same manner.

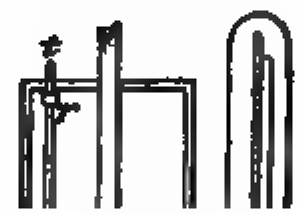
The working of the piston of the pump, is opposed by the pressure of the atmosphere; hence another pump barrel has been added, and the two pistons made to act alternately, the one rising as the other falls. By this method, the pressures on the pistons are made to oppose each other. To work the pistons with greater facility, their rods are cut into teeth, forming racks: between these, and interlocking with both, is placed a pinion, or toothed wheel. The latter is turned by a winch, and sometimes by a double handle.

The resistance of the atmosphere may be in a great degree removed, by making the cylinder of the pump air tight; the piston rod must, in this case, work through a collar of leathers; a lateral spout is therefore placed to permit the escape of the air, and this contains a third valve, opening upwards. A pump of this structure works with greater ease, as the exhaustion of the receiver increases, while in those with open barrels, the resistance augments with the exhaustion.

In the earlier pumps, both the valves were opened by the elasticity of the air; and thus a limit to the exhaustion was attained, in the resistance of the valve to an effort to open it. In order to increase the power of the pump in exhaustion, the lower valve is now made to open by the rise of the piston, and is shut by its descent.

Such an arrangement is represented on the succeeding page, being a section of the air pump of Demoutier.

P P′ is the piston in which is seen the upper valve *s*; the lower valve is represented at *c*; it has a conical form, and is attached to



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a rod *c t*, that passes through the piston, and its leather packing, with so much friction as to be air tight. As soon as the piston begins to rise, the rod *c t* will, in consequence of the friction, move with the piston, and the valve *c* opens; but the rod moves through no more space than is just necessary to open the valve, being checked by a shoulder at *r*, that strikes against a plate that forms a lid to the barrel of the pump.

The valves of the pumps that first succeeded that of Guerické, were made of strips of bladder, or of oiled silk: in the pump just represented, the upper valve is of steel, resting on an oiled leather seat; the lower valve is a leather conic frustum, applying itself to a hollow frustum formed in a steel plate. In the best English pumps, the valves and seats are both of metal, and the form that of a conic frustum.

We have stated that oil and brass mutually act upon each other; hence, the barrels and pistons that were originally made of brass, and rendered air tight, by packing of leather soaked in oil, were rapidly worn. To remedy this defect, the pistons are now made of steel, which is effectually preserved from rust by the oil used in the packing, and the barrels of the best pumps are made of glass.

360. The power of air pumps, to exhaust the receivers, placed upon their plates, is measured by means of instruments called Gauges. These are of several kinds.

(1). The Barometer Gauge.—This consists of a glass tube about 30 inches in length, and open at both ends; it is placed in a vertical position attached to the stand of the pump; the lower end is plunged in a basin of mercury; the upper is cemented air tight

to a pipe that leads to the opening in the plate of the pump, by which the air passes from the receiver; or it may communicate in some other manner with the valves of the pump. As the air in the receiver is exhausted, the pressure of the external atmosphere, forcès the mercury in the basin to rise in the glass tube. The difference between the height to which the mercury is thus raised, and that at which it is supported at the same time in the sealed Torricellian tube, or Barometer, is obviously a measure of tension of the air yet remaining in the receiver. We shall hereafter see that this tension is exactly proportioned to the quantity of air that remains.

(2). A tube 6 or 8 inches in length, sealed at one end, and filled with mercury, is inverted in a basin of the same liquid. The mercury is of course supported by the pressure of the atmosphere, and continues to fill the tube. This apparatus is placed beneath a receiver, communicating with the valves of the pump. During the first stages of the exhaustion, the mercury still remains supported, but so soon as the tension of the contained air becomes less than is sufficient to support such a column of mercury, that liquid begins to fall in the tube; the height at which it stands above the level of that in the basin, is a measure of the tension of the remaining air. This is called the Short Barometer Gauge.

(3). The Syphon Gauge acts upon the same principle as the last; but instead of plunging the open end of the tube in a basin of mercury, it is bent upwards; the whole is therefore composed of two parallel branches, one open, and the other closed. The latter is filled, before the action of the pump begins, with mercury.

These two gauges are placed beneath separate receivers, expressly adapted for them to an additional part of the pump, in order that they may be used at the same time that other receivers are placed upon the principal plate.

(4). The Pear-Gauge — a vessel of glass of the shape denoted by its name, having a small opening at the larger end, is sunk by means of weights into a basin of mercury, with its opening downwards. This apparatus being placed beneath a receiver, and the pump set in action, the air contained in the pear-shaped vessel will expand, and make its way through the mercury: thus, the exhaustion within it will correspond with that effected in the receiver. On re-admitting the air to the receiver, the pressure upon the mercury in the basin will force it into the pear-shaped vessel, until the portion of air left within it be restored to about its original density. Hence, the relation between the part of the vessel not filled by the mercury, and its whole capacity, will be a measure of the exhaustion of the pump.

When this gauge is used with a pump, to whose plate the receiver is adapted by means of a sheet of leather soaked in oil, it will exhibit a far higher degree of exhaustion than any of the three preceding gauges. This grows out of the fact that the elastic vapour of the oil, which affects them, cannot enter it. This observation led to the improvement we have mentioned, of fitting the receivers to the plate of the pump by grinding.

361. The air pump affords a great variety of proofs of the pressure of the atmosphere, of which a few may be cited:

(1). In the manner in which the receiver is fixed to the plate of the pump, in such a way as to act as a close vessel; in its remaining firmly attached to the plate after the air is exhausted.

(2). The hand placed upon an open receiver, will in like manner be firmly fastened to it; and the pressure which was before equal on both sides of the hand, and therefore imperceptible, will become painful.

(3). A square flask of glass, screwed to the opening in the plate of the pump, will be broken, as will a glass plate, ground to fit an open receiver.

(4). A small receiver, suspended by means of a rod passing through a collar of leathers, within a larger receiver, may be depressed by means of the rod, till it rests on the plate of the pump. If this small receiver do not cover the orifice by which the plate communicates with the valves, the re-admission of air into the larger receiver, fixes the smaller to the plate of the pump; nor can it be removed until the air be again exhausted.

(5). Two hemispheres of brass ground to fit each other, may be affixed to the plate of the pump by means of a pipe proceeding from one of them, and which is screwed to the orifice. If the air be exhausted, they will be found to be pressed powerfully together. If there be a stop-cock on the pipe of communication, it may be closed, and the apparatus detached. The spheres held together by the pressure of the atmosphere may be then separated by means of weights; one of them being suspended, and the weights attached to the other. The quantity of weight required to separate them, will be a measure of the pressure they sustain: as the surface of a sphere of known diameter can be calculated, this was used as a measure of the atmosphere's pressure upon each unit of square measure. It is, however, affected by the air which remains in the apparatus, and which the pump cannot wholly exhaust; it is, therefore, inferior in accuracy to the tube of Torricelli.

Experiments for illustrating the pressure of the atmosphere by means of the air pump, may be multiplied to a great extent, but the above are sufficient for our purpose.

362. That the pressure of the atmosphere is the cause of the rise of fluids in pumps, and of mercury in the Torricellian experiment, may also be shown by means of the air-pump. Thus, if a model of a pump be passed through a collar of leathers, adapted to the top of a receiver, it will be found to be without effect, when the reservoir in which it is plunged is not pressed by air. So if a barometer be in like manner inserted into the top of a receiver, the mercury will fall in the tube as the air is exhausted from the receiver.

363. To show the weight of air, a flask furnished with a stopcock is weighed; it is then screwed to the plate of the pump, and the air exhausted; the stopcock being closed, it may be removed without admitting air, and being again weighed, it will be found lighter than before. To obtain the exact weight of a given bulk of air, demands certain precautions which will be hereafter stated.

364. The elasticity of the air is manifest from the manner in which the air-pump itself works; it may also be exhibited in various other manners, thus:

(1.) If a bladder, partially inflated with air, be placed under a receiver, and the air withdrawn from the latter, the bladder swells, and is distended; on re-admitting the air to the receiver, the bladder collapses to its original dimensions.

(2.) If the bladder, when partially filled, be loaded with weights, and placed beneath a receiver, the contained air will, in expanding itself as the receiver is exhausted, lift the weights.

(3.) If a glass mattrass be inverted in a vessel containing a liquid, and placed beneath the receiver of an air-pump; as the air is exhausted from the receiver, that confined in the mattrass by the liquid will expand, and escape in visible bubbles through the fluid mass. On re-admitting the air, the liquid will be forced up the neck of the mattrass, and into the bulb, which it will, if the exhaustion have been sufficient, nearly fill. On a second exhaustion, the air will expand, as will be manifested by the descent of the liquid confined in the mattrass; and the latter will be forced back a second time on the re-admission of air.

CHAPTER VII.

EQUILIBRIUM OF PERMANENTLY ELASTIC FLUIDS.

365. Although the elasticity of the air is demonstrated by the experiments cited at the close of the last chapter, they do not exhibit the law of that elasticity ; nor do they point out what relation the force, with which it expands itself, bears to the force by which it is compressed. This is a question whose importance demands a particular investigation, which may be performed by the following experiments.

(1.) Take a glass vessel with a narrow neck, (one nearly of a spherical figure is best adapted for the purpose), and pour into it a small portion of mercury; if we then screw, air-tight, to its neck a slender tube of glass open at both ends, of the length of at least thirty inches, whose lower end is wholly immersed in the mercury, the air that fills the upper part of the vessel will be wholly separated from the general mass of the atmosphere. If this apparatus be placed beneath a tall receiver, upon the plate of the air pump, and the air exhausted; the pressure on the air, confined in the vessel, will be lessened in proportion as the tension of the air in the receiver is diminished; the elastic force of the confined air will, in consequence, cause a column of mercury to rise in the tube, and the height of this column will be a measure of the difference between the tension of the air in the receiver, and that of the air of its original density confined in the vessel. If the pump have a long barometer gauge (No. 1 of § 357), the mercury that rises in it furnishes a similar measure of the difference between the pressure of the external atmosphere and the tension of the air remaining in the receiver.

The relation between the two columns of mercury furnishes a direct comparison between the pressure of the atmosphere, and the elasticity of the air that composes the part on which that pressure takes place; for the confined air has obviously been enclosed under that very pressure. On comparing the heights of these two columns, during all the periods of the exhaustion, they will be found to all appearance exactly equal, the mercury rising in the two tubes *pari passu*. The small change in the bulk of the confined air, growing out of the mercury it forces out of the vessel, is insensible. Hence, it is obvious that the elasticity of air, at the ordinary density of the atmosphere, is exactly equal to the pressure.

(2.) To examine the effect of increased pressure, we take a cylindrical glass tube bent into two branches; one of these is closed, and the other open. Mercury is then poured in until it fill the bend of the tube; a portion of the air that would be thus confined in the closed branch is permitted to escape, by inclining the apparatus, until the mercury stands at the same level in both branches of the tube. The confined air has then no other pressure to sustain than that of the atmosphere, for the columns of mercury in the two branches counterbalance each other. The air, therefore, that is confined, has, as may be inferred from the preceding experiment, the same density as that of the adjacent stratum of the atmosphere. In order to increase the pressure, mercury is poured into the open branch of the tube; the confined air is then affected by a pressure equal to the sum of that of the atmosphere, and that of the column of mercury measured from the level at which it stands in the close branch of the tube.

In this experiment, the level of the mercury in the closed branch of the tube will be found to rise as the pressure caused by the column in the open branch increases, thus marking a condensation in the enclosed air. By the uniform result of all experiments, after employing the precautions that will be presently stated, it is found that the space occupied by the confined air, is inversely as the pressure to which it is subjected.

In order to express this fact, let B be the original volume of the air, under the pressure, p, namely that of the atmosphere; B′ a volume under the increased pressure, p';

$$\frac{B'}{B}=\frac{p}{p'};$$

whence we obtain

$$B'=\frac{Bp}{p'}; \qquad (375)$$

and for the volume B″, under any other pressure p'',

$$B''=\frac{Bp}{p''};$$

dividing by B′ we obtain

$$\frac{B''}{B'}=\frac{p'}{p''};$$

whence

$$B''=\frac{B'p'}{p'}. \qquad (376)$$

By which we may calculate the relation the volumes of the same mass of air under any given pressures whatsoever.

The precautions to which we have referred, grow out of the following circumstances:

(*a*) Air in being compressed has its temperature raised; hence the apparatus must be pérmitted to cool down to the temperature of the surrounding air before the observation is made.

(*b*) All air contains a greater or less proportion of aqueous vapour, and this, as we shall presently see, is not affected by pressure in exactly the same manner as air, under similar circumstances. In order to make the experiment perfectly satisfactory, the tube must be well dried, and the moisture withdrawn from the air it contains, by exposing the latter to the contact of hygrometric substances, for some hours before the experiment is made.

The space being found, after these precautions have been taken, to be inversely proportioned to the pressure, it may be inferred that in the case of the condensation of air, such as is usually found at the surface of the earth, the density is always proportioned to the pressure; and to this, the tension, or elastic force, is equal.

(3.) It remains that the law of elasticity should be determined for air of diminished density. For this purpose, take the Torricellian apparatus and fill the tube partly with mercury; the remaining part will continue to contain air. On placing the finger upon the open end, and inverting the tube, the air will rise through it to the close end, and will, so long as the finger is tightly pressed on the opening, occupy the same space it did at first. But on immersing the open end of the tube in a basin of mercury, the confined air is no longer compressed by the whole force of the atmosphere, for the latter must also support the column of mercury. The confined air then, being less compressed, expands itself, and causes the mercury to descend; and it finally comes to rest in such a position that the sum of the pressure of the column of mercury, and of the tension of the confined air, is equal to the pressure of the atmosphere.

To ascertain the law from this experiment, let f be the tension of the air after its expansion; p, the pressure of the atmosphere; b, the original volume of the air, and x, that to which it expands itself; it is found that

$$x=\frac{pb}{f};$$

whence we obtain

$$\frac{x}{b}=\frac{p}{f}, \tag{377}$$

which is identical with the formula (375); hence the law is the same, in the case of rarefied as in that of condensed air.

The same experiments may be performed with dry gases, and the results are found the same as when they are performed with atmospheric air. Hence, the air of the atmosphere and all other

dry gases, at constant temperatures, occupy spaces that are inversely as the pressures to which they are subjected. This law was originally discovered by Mariotte, and goes by his name. It is true at all mean and usual pressures, but ceases to be so at certain limits. Were it absolutely true, the smallest possible quantity of air would, on the pressure being wholly removed, occupy a space infinitely great; while there could be no space so small into which the largest mass of the air could not be compressed by a sufficient force. The limit is found, on the one hand, in the fact that nearly all the gases have been condensed into the liquid form; and even atmospheric air, as is stated by Perkins, has been reduced to that state. The limit, on the other hand, appears rather to arise from the relation of expanding air to heat; for in the higher regions of the atmosphere, the expansion produces a cold so intense, that the diminution of temperature will finally produce an effect equal and contrary to that caused by the removal of pressure. Laplace, in stating this limit, has expressed the opinion: that although the existence of the limit, and consequently the finite extent of the atmosphere, is capable of demonstration; the exact height at which it ceases to expand is not within the reach of calculation. We are therefore compelled to have recourse to a physical fact to estimate the probable extent of the atmosphere of the earth; this is the phenomenon of twilight, whence it is inferred, that the air of the atmosphere still retains a sufficient density to reflect the rays of light, at a distance of forty miles from the surface of the earth.

Were atmospheric air capable of indefinite expansion, the mass that surrounds the earth would have distributed itself around the earth and moon, in the ratio of their respective masses, by the influence of the attraction of gravitation; and generally, had any one planet, at the time of the creation, been surrounded by an atmosphere of such a character, all would have derived from it atmospheres distributed in a similar ratio. Now the moon has no perceptible atmosphere, and we have no reason to conclude that the other bodies of the solar system have atmospheres, of the density and mass that might have been inferred from the law of Mariotte: hence, again, we find a corroboration of the opinion, that the atmosphere of the earth has a finite extent. The same was inferred by Wollaston, from astronomic observations, into the detail of which it is not our province to enter.

366. We may now proceed to investigate the conditions of equilibrium in the air that composes our atmosphere, under the supposition that it is of uniform temperature throughout.

If we use the notation of § 316, the density s being, by the law of Mariotte, directly proportioned to the pressure, p,

$$s=mp; \tag{378}$$

in which equation, m is a constant co-efficient to be determined by experiment.

The equation (352),

$$dp=s\,dz,$$

will give

$$mz=\log. p;$$

and if c be the modulus of the tables,

$$p=c^{mz}.$$

If, as is most convenient, we conceive the origin of the co-ordinate z, to be at the mean surface of the earth, or at the level of the sea,

$$dp=-s\,dz; \tag{379}$$

and substituting the value of s,

$$dp=-mp\,dz. \tag{380}$$

If P be the pressure of the atmosphere at the level of the sea, when $z=0$, $p=\mathrm{P}$; and integrating the foregoing equation, we obtain

$$mz=\log. \mathrm{P}-\log. p, \tag{381}$$

and

$$p=\mathrm{P}c^{-mz}. \tag{382}$$

Hence:

The altitude of any point in the atmosphere, above the level of the sea, is proportional to the difference between the logarithms of the respective pressures; and the difference of level between any two points in the atmosphere, is in a similar manner proportioned to the difference of the logarithms of the pressures at these two points. The column of mercury in the Torricellian tube is the measure of the pressure at the point where it is placed, and therefore the difference in the logarithms of the height of the two columns, at the two points, is in like manner proportioned to their difference of level.

In the equation (380), if $p=0$, z becomes infinite; hence we reach the conclusion already stated, of the infinite extent of the atmosphere, when the relations of air of different densities to temperature is not taken into account.

367. If the temperature be not constant, as it is not in practice, it becomes necessary to take into view the dilatation and contraction of the air by changes of temperature. We owe the original determination of the law of the dilatation of air to Gay Lussac. Similar inferences were obtained separately, at about the same

period, by Dalton. Still more accurate experiments have since been performed by Petit and Dulong. It is not our province to enter into the detail of this inquiry, but it is sufficient to express the general law, which is:

All permanently elastic fluids expand an equal proportion of their volume, estimated at some given temperature, for equal increments of temperature, and this proportion is the same in them all. Within the usual limits at which experiments or observations are made, the expansion of mercury in a glass tube, or its rise in the thermometer, is also uniform: hence, every permanently elastic fluid expands an equal and constant quantity for every degree of thermometric heat, and this is usually estimated in the form of a fraction of its bulk at the temperature of melting ice. The experiments of Petit and Dulong make this fraction 0.002083 for every degree of Fahrenheit's thermometer.

368. The two laws of Mariotte and Gay Lussac combined, will enable us to determine the density of atmospheric air, under any pressure and at any temperature, its density at some given pressure and temperature being first known.

If a be the constant co-efficient of dilatation, 0.00208; t the number of degrees of the thermometer above or below 32°; any given unit of bulk of air, under the mean pressure of the atmosphere, and at the temperature of 32°, will become at the temperature t,

$$1 \mp at;$$

and if P be the mean pressure of the atmosphere, and P′ any other pressure, this bulk will, by the law of Mariotte, become

$$\frac{(\pm at)P}{P'}. \tag{383}$$

As the volumes are inversely as the densities, if D be the density under the pressure P, and temperature 32°; and D′ the density at the pressure P′, and temperature t,

$$D : D' :: \frac{(\mp at)P}{P'} : 1;$$

whence

$$D' = D\frac{P'}{(1 \mp at)P}. \tag{384}$$

And so in respect to any other gas, whose density at the mean pressure P, and temperature 32°, is D″, we obtain for its density D‴, at the temperature t, and pressure P′,

$$D''' = D''\frac{P'}{(1 + at)P}; \tag{385}$$

and by comparison with the last formula, (384),

$$\frac{D}{D''}=\frac{D'}{D'''}. \qquad (386)$$

Thus the densities of gases preserve the same relation to each other at every temperature and every pressure.

369. By reference to the investigations of Chapter III., and comparing them with the characters of elastic fluids, it will be seen that they possess, in some respects, properties in common with liquids. Thus:

(1.) The pressure of an elastic fluid will be proportioned to the surface on which it presses, and is measured by the weight of a prism of the fluid, of uniform density throughout, whose base is equal to the surface pressed, and whose altitude is such, that if the fluid were of uniform density, the pressure on the unit of surface would be the same as it is found to be under the law of variable density.

(2.) A solid body, immersed in an elastic fluid, loses as much weight as is equal to the weight of the fluid it displaces, or of a volume of the fluid equal to its own. Hence the intense pressure of 15 pounds on each square inch of the surface of our bodies, and which on the body of a middle-sized man, amounts, if the mere intensity of the pressures be estimated, without reference to their counteracting directions, to 11 tons, has for its resultant, § 336, only the weight of a mass of air equal to our bodies in volume; and as the direction of this resultant is upwards, so far from being crushed by this pressure, we are actually supported by it. If, however, the pressure be removed from any part of our bodies, it becomes manifest on the rest, and presses the part to which the exhaustion is applied against the apparatus that operates the exhaustion. Such is the case of the experiment, § 361, No. 2.

This buoyancy of the air will render the determination of the specific gravity of bodies, by the methods of Chap. IV., inaccurate; for instead of their absolute weight, we obtain the weight less that of an equal bulk of air. A correction may, however, be applied to the approximate specific gravity, which rests on the following principles.

Let w be the absolute weight,
W' the weight in air,
W the weight in water,
S the specific gravity as usually obtained,
s' the true specific gravity,
m the specific gravity of air.

The absolute weight will be greater than the weight in air by the weight of a mass of air equal in bulk to the body. As it should

lose in water $w—W'$, it will lose in air $m\,(w—W')$, we have for the value of the absolute weight,

$$w = W' + m\,(w - W').$$

We may, without sensible error, substitute W for w, in the quantity $w—W'$, and the equation becomes

$$w = W + m\,(W - W').$$

We may then apply the correction $m\,(W—W')$ to the weight in air, before dividing by the loss of weight; and the latter may be taken in reference to this corrected weight instead of the weight in air.

The value of m will be determined hereafter.

The exact value of the absolute specific gravity will be

$$s = S + m\,(1 - S).$$

CHAPTER VIII.

OF THE EQUILIBRIUM AND TENSION OF VAPOURS.

370. Water, which boils under the mean pressure of the atmosphere at 212°, boils under diminished pressures at a lower temperature, and may, by increased pressures, be prevented from boiling until it attain a greater degree of sensible heat. The same is the case with other liquids; each has a determinate degree of temperature at which it boils, under the mean pressure of the atmosphere; and the boiling point is raised or lowered, with the increase or diminution of the pressure to which it is subjected.

In the ebullition, an elastic fluid is generated, and it is only when the tension of this elastic fluid becomes equal to the pressure, that the action called boiling, takes place. These elastic fluids may, by the abstraction of their latent heat, be restored to the liquid form, and hence belong to the subdivision we have styled vapours.

Although that rapid formation of vapour, which is attended with ebullition, takes place under a given pressure, at a constant temperature in each different liquid; still the liquid is capable of rising in the form of vapour at lower temperatures. Thus, although water, under ordinary circumstances, does not boil below 212°, yet if left exposed in an open vessel, a gradual diminution of its bulk will take place at lower temperatures, and it will finally be wholly dissipated. This has been conclusively shown by Dalton, to be owing to the formation of vapour of a temperature identical with that of the water whence it proceeds; and of an elastic force, identical with that which would proceed from water, boiling at that temperature, under diminished pressure. Even ice is capable of furnishing vapour of its own temperature, and an appropriate tension. The same is the case with other liquids; although all boil under given pressures, at some temperature constant in each; all give out vapour insensibly at less elevated temperatures.

371. To measure the tension of aqueous vapour, a portion of water may be introduced into the Torricellian apparatus, where it will float upon the surface of the mercury. Vapour of the temperature of the water, will immediately form in the vacuum. Its pressure will be indicated by its causing the mercury to descend from its original height; the tension of the vapour will be mea-

sured by the space through which the mercury falls; or by the difference in the height of the mercury in the tube in which the experiment is performed, and that at which it stands in the barometer.

If the tube be of considerable length, and if, instead of a basin, another tube of larger diameter than the former be used to hold the mercury in which it is inverted, the pressure upon the vapour contained in the Torricellian vacuum, may be varied. Thus: By raising the inverted tube the pressure may be diminished; in this case it is found that so long as any water remains, new vapour of the same tension will be generated, and the altitude of the mercury in the tube will remain constant. If the tube be depressed, the increase of pressure causes the condensation of a portion of the contained vapour, and the mercury will almost instantly rise to its original height above the surface of the mercury in the outer tube; or if the depression be performed slowly, the mercury will move up the inner tube exactly in proportion as the latter is pressed down, and will stand at a constant altitude.

If the quantity of water in the tube be so small that in raising the tube the whole will be converted into vapour, it will be found that after that time the tension of the confined vapour diminishes, as does that of air, with the increase of the space it occupies.

It may hence be inferred that the law of Mariotte is inapplicable to aqueous vapour, so long as it is in contact with the liquid whence it is generated; but this law becomes true at diminished pressures, when no water is present. The same is found to be true in the case of all the vapours, on which experiment has been made. All vapours then, have, at a given temperature, a maximum of density and tension that cannot be exceeded; and on attaining which, they are condensed into the liquid form by any attempt to compress them.

The essential characteristic of a vapour, therefore, is, that at a given temperature, no more than a limited quantity by weight can exist in a given space; so that by diminishing the space, the whole excess is reduced by pressure to the liquid form, without any increase taking place in the tension. While, when the space that gases occupy is diminished by pressure, their tension, at a constant temperature, is increased exactly as the space they occupy is lessened.

372. It may be easily ascertained in the course of these experiments, that the tension of the vapour is increased by an increase of temperature, and *vice versa*. We shall hereafter inquire into the law that this increase follows. If then vapour be formed, or admitted into a space unequally heated, it might at first sight appear that the vapour would have unequal tensions. This, how-

ever, is not the fact, for in consequence of its elasticity and fluid nature, it would tend to an equilibrium of tension; the warmer portions passing to the parts of the space that are less heated, would have their temperature lowered, and tension diminished; and thus, in a space unequally heated, the maximum of tension, when equilibrium is established, is no more than what is due to the minimum temperature.

373. When a vapour, after being formed, is heated out of contact with any of the liquid whence it is generated, it tends to expand, and its elastic force is increased exactly as if it were a gas.

Hence, at a constant density, its pressure becomes, § 368,

$$P' = P\,(1 \mp at); \tag{387}$$

and under a constant pressure, its density is, § 368,

$$D' = \frac{D}{(1 \mp at)}. \tag{388}$$

But when a vapour is heated in contact with the liquid whence it is generated, the latter acquires the power of furnishing vapour until the maximum of tension due to that temperature is attained. Thus not only does the expansive force increase with the temperature, according to the law of Gay Lussac, but with the density caused by the new evaporation, according to the law of Mariotte.

The relation between the temperatures, densities, and tensions of vapours might, therefore, seem susceptible of determination.

In effect, in the case of water which boils at 180° above the freezing point, an investigation analogous to that of § 367, gives us,

$$D' = D\,\frac{P'\,(1+180\,a)}{P\;\;(1+at)}. \tag{389}$$

We are however unable by this, to determine the tensions independently of the densities, and are, therefore, compelled to have recourse to experiment, in order to determine the elastic force of steam and other vapours at given temperatures. From these experiments empirical formulæ may be deduced.

374. The best experiments on the tension of aqueous vapour at low temperatures, are those of Dalton, which are comprised in the following table.

TABLE

Of the Maximum Tension of Aqueous Vapour, at temperatures below 212°, estimated in the height of the column of Mercury they are capable of supporting.

Temperature.	Tension in inches of Mercury.	Temperature.	Tension in inches of Mercury.
2°	0.068	117°	3.08
12°	0.096	122°	3.50
22°	0.115	127°	4.00
27°	0.168	132°	4.60
32°	0.200	137°	5.29
37°	0.237	142°	6.05
42°	0.283	147°	6.87
47°	0.339	152°	7.81
52°	0.401	157°	8.81
57°	0.474	162°	9.91
62°	0.560	167°	11.25
67°	0.655	172°	12.72
72°	0.770	177°	14.22
77°	0.910	182°	15.86
82°	1.07	187°	17.80
87°	1.24	192°	19.86
92°	1.44	197°	22.13
97°	1.68	202°	24.61
102°	1.98	207°	27.20
107°	2.32	212°	30.00
112°	2.68		

From the numbers in this table the following formula for the elastic force P, has been deduced, in terms of degrees of the temperature t, reckoned from 32°,

$$P = .1781(1 + .006\,t)^7. \tag{390}$$

A formula better adapted to atmospheric temperatures, is

$$P = .18 + .007\,t + .00019\,t^2 \tag{391}$$

The tension of aqueous vapour or steam, at temperatures above the boiling point, is usually estimated in atmospheres. The latest experiments are those of Dulong and Arago, which give the results of the following table, as far as 24 atmospheres from observation, and as far as 50 from calculation.

The formula used in the calculations, and which is deduced from the observations, is

$$P = (1 + .00397\,t)^5; \tag{392}$$

t being estimated from 32°.

TABLE

Of the Elastic force of Steam at corresponding temperatures, from 1 to 50 Atmospheres.

Tension in Atmospheres.	Corresponding Temperature.	Tension in Atmospheres.	Corresponding Temperature.
1	212°	9	350°.8
1½	229°.2	10	358°.9
2	250°.5	11	366°.8
2½	263°.8	12	374°.
3	275°.2	13	380°.6
3½	285°.1	14	386°.9
4	294°	16	398°.5
4½	301°.3	18	408°.9
5	308°.8	20	418°.5
5½	314°.	22	427°.3
6	320°.4	24	435°.6
6½	326°.5	30	457°.2
7	331°.7	40	486°.6
8	341°.8	50	508°.6

375. In experimenting on the tensions of the vapours of substances other than water, Dalton discovered that a remarkable law held good within the limits at which his experiments were made. This law goes by his name, and is as follows, viz:

Every different liquid has a determinate temperature at which it boils under the mean pressure of the atmosphere. At this temperature the elastic force of its vapour is just equal to the pressure of the atmosphere. At other temperatures equidistant from the boiling points of the different liquids, their respective tensions are still equal.

Thus: water boils at 212°, and ether at 96°, the tension of the vapours being in both cases the same; when water is heated under pressure to 250°.5, its tension is doubled, for an increase in temperature of 38°.5; and so, the vapour of ether at the temperature of $96°+38°.5=134°.5$, has an elastic force equivalent to two atmospheres.

376. By an apparatus similar in principle to the tube of Mariotte, § 365, (2) air may be enclosed, and subjected to any given pressure. If this air be perfectly dry, and if water be passed up into the space occupied by the air, it will be found that it evaporates at all temperatures whatsoever, furnishing a vapour whose tension may be determined by the change it causes in the column of mercury that confines and compresses the air. The rapidity with which this vapour is given out, will be affected by the pressure; being instantaneous, as has been shown, in vacuo, and becoming less rapid for every increase in the pressure.

After a time, however, the pressure of the vapour adds, to the original tension of the confined air, an elastic force which is exactly equivalent to the maximum tension of vapour of the temperature at which the experiment is made. The evaporation then ceases, and the joint tension of the confined air and vapour remains constant.

Thus the presence of air does not effect the maximum tension of vapour of a given temperature, but merely retards its formation; and the quantity of aqueous matter in the elastic form that can exist in a given space is the same, whether that space be void of any other substance, or filled with air of any density whatsoever. Vapour then, like a gas, tends to distribute itself around the earth in an atmosphere, and the formation of such an atmosphere is not prevented, but merely retarded, by the presence of an aeriform atmosphere.

Vapour when mingled with a permanently elastic fluid, may be condensed by the same two causes that induce its precipitation in vacuo: namely, by an increased pressure, and by a diminished temperature.

The laws that are applicable to the mixture of two gases, or of a gas with a vapour, are true of the mixture of any number of elastic fluids, whether they be permanently so or not. And thus in a given space, any number of elastic fluids whatever may be enclosed, without these substances interfering with each other. It is only necessary that they be subjected to a pressure equal to the sum of their several tensions.

CHAPTER IX.

Of the Specific Gravity of Elastic Fluids.

377. We may, as has been stated in § 363, obtain the weight of a mass of atmospheric air, by taking a flask furnished with a stop-cock, weighing it, and then adapting it to the plate of the air-pump to exhaust the air. The stop-cock being closed, the flask may be removed, and again weighed; the difference between its weight under the two different circumstances, is the weight of the air that has been withdrawn.

As the air of the atmosphere always contains moisture, a more correct measure of the weight of the air the flask is capable of containing, may be obtained, by taking the last weight as the absolute weight of the flask; air that has been carefully dried by hygrometric substances is then introduced, and the flask again weighed. We thus obtain its weight, free of the influence of the vapour it contains under ordinary circumstances.

The weight of any other gas may be ascertained in a similar manner, by introducing it, after being well dried, into a flask, whence the air has been previously exhausted.

In performing these experiments, it will be obvious that the quantity of gas that will enter the vessel, depends upon the pressure at which it is introduced, and upon its own temperature; for under different pressures it will have different densities, the temperature remaining constant; and at different temperatures the density will also vary, the pressure remaining constant. Hence, both the temperature and pressure, at which the experiments are made, must be noted.

The flask in which the experiment is made, will also vary in size, under the influence of temperature; the effect thus produced, in consequence of the dilatability of the glasss, must therefore be taken into account.

If the gas cannot be introduced perfectly dry, its hygrometric state must be observed.

The operations of weighing are performed in the air of the atmosphere; hence, the apparent weight of the flask, whether full or empty, will be less than the true by the weight of an equal volume of air. This loss of weight will be affected by the pressure and temperature of the air, and by the quantity of aqueous vapour it contains.

The air-pump does not exhaust the whole of the air from the flask, and hence the proportion that remains, which will be indicated by the guage of the pump, must be noted.

Such are the principles on which the determination of the weight of atmospheric air, and different gases depend. The detail of the operations, and the manner of applying the corrections may be seen by reference to Biot: *Traité Complet de Physique, Vol. I.*

The capacity of the flask being known, the weight thus obtained may be compared with that of an equal bulk of water, and the specific gravity determined upon the principles of Chap. IV, in terms of that liquid as a unit. It has been thus found that the specific gravity of atmospheric air, at the temperature of 32°, is 0.001299055. Its density at any other temperature and pressure, may be obtained by means of the formula (364). This fraction is the value of m, in the formula for the absolute specific gravity of bodies, § 369.

This comparison is necessary, in order to connect the tables of the specific gravities of gases with those of solid and liquid bodies, in the latter of which water is employed as the unit of density.

When the specific gravities of gases are sought, it is usual to determine them in terms of some body of that class, taken as the unit. Atmospheric air, which, when freed from moisture, has an uniform constitution in all parts of the globe, is most convenient for this purpose. It is, therefore, most frequently employed in mere mechanical investigations. But for many purposes in chemistry, hydrogen possesses superior advantages, particularly from the fact of the numbers that represent the densities being whole, in consequence of the great levity of that gas. Some chemical writers employ oxygen as the unit.

We subjoin a table of the specific gravities of some of the gases, in terms of atmospheric air.

TABLE

Of the Specific Gravities of Gases.

Atmospheric Air . .	1.0000
Hydriodic Gas . . .	4.4288
Silico-Fluoric Gas . .	3.5735
Chlorine	2.4216
Sulphurous Acid Gas . .	2.1930
Cyanogen . . .	1.8064
Nitrous Oxide . . .	1.5269
Carbonic Acid Gas . .	1.5245
Muriatic Acid Gas . .	1.2474
Sulphuretted Hydrogen .	1.1912
Oxygen	1.1026
Nitrogen	0.9757

Carbonic Oxide . .	0.9569
Ammonia . . .	0.5967
Carburetted Hydrogen .	0.5596
Hydrogen . . .	0.0688

378. The density of vapour may be determined by introducing a known weight of the substance that yields it, into a receiver containing mercury, supported by the pressure of the atmosphere, and inverting the receiver in a vessel also containing mercury, upon the principle of the Torricellian experiment. The outer vessel is tall enough to contain a mass of some transparent fluid of sufficient depth to cover the receiver. The whole apparatus is then heated; and when the whole of the substance, whose vapour is under experiment, has been evaporated, the space the vapour occupies and its temperature are noted. We then have a bulk of the vapour, whose weight is the same as that of the substance whence it was generated; the temperature is known, and the pressure can be determined by means of the columns of mercury and of the surrounding liquid compared with the indication of the barometer at the time. In this manner, it is found that the density of steam at the temperature of 212° is $\frac{1}{1694}$ part of the density of water at 32°. The densities of vapours compared with atmospheric air as the unit, are as follows:

TABLE

Of the Specific Gravities of the Vapours of Different Substances, at their boiling temperatures, under the mean pressure of the atmosphere. Atmospheric air at the temperature of 32°, and under the same pressure, being taken as the unit.

Vapour of Iodine . . .	8.6111
of Oil of Turpentine .	5.0130
of Nitric Acid . .	3.1805
of Sulphuret of Carbon .	2.6447
of Sulphuric Ether .	2.5860
of Pure Alcohol . .	1.6133
of Water . . .	0.6235

379. The density of a vapour, under a given pressure, and at a given temperature being known, that under any other pressure, and at its corresponding temperature, can be calculated by the formula (389). In this manner the density of steam, in terms of water, and the relative volumes occupied by given weights of aqueous vapour, have been calculated. The pressures, and the temperatures, above the boiling point, have been taken from older experiments than those of Arago and Dulong. They have been purposely retained in order to show the view of this subject that is still most generally received.

TABLE

Of the Density and Volume of Aqueous Vapour, at its Maximum Tension. The Unit of Density being water at the temperature of 32°, *and that of volume, the volume of an equal weight of water, also at* 32°.

Temperature.	Pressure in Atmospheres.	Density.	Volume.
32°	"	0,0000053	188600
41°	"	0,0000073	137000
50°	"	0,0000097	103000
59°	"	0.0000131	76330
68°	"	0.0000173	57800
77°	"	0.0000227	44050
86°	"	0.0000297	33670
95°	"	0.0000390	25640
104°	"	0.0000499	20030
113°	"	0.0000637	15690
122°	"	0.0000710	14080
131°	"	0.0001022	9784
140°	"	0.0001261	7930
149°	"	0.0001592	6281
158°	"	0.0001964	5091
167°	"	0.0002388	4187
176°	"	0.0002936	3406
185°	"	0.0003557	2811
194°	"	0.0004261	2346
203°	"	0.0005074	1971
212°	1	0.0005900	1696
251.6°	2	0.00110	909
291.2°	4	0.00210	476
330.8°	8	0.00399	250
370.4°	16	0.00760	131

We may, by means of this table, calculate the weight of water that would be contained in the form of vapour of the maximum tension, at a given temperature, in some unit of cubic measure. Such tables are useful in the consideration of atmospheric phenomena; and as the air rarely or never contains mixed with it the greatest weight of water that can exist at that temperature, it is also important to know the rate at which vapour expands at each temperature. These are combined in the following table, extracted from that of Daniell.

TABLE

Of the Force, Weight, and Expansion of Aqueous Vapour, at different degrees of temperature, from 0° to 92°.

Temperature.	Elastic force in inches of mercury.	Weight of a cubic foot in grains.	Expansion.
0°	0.064	0.789	1.000
12°	0.096	1.156	1.024
22°	0.139	1.642	1.045
32°	0.200	2.317	1.066
42°	0.283	3.214	1.087
52°	0.401	4.468	1.108
62°	0.560	6.126	1.129
72°	0.770	8.270	1.150
82°	1.070	11.293	1.170
92°	1.440	14.931	1.191
212°	30.000	257.191	1.441

380. The presence of aqueous vapour in atmospheric air, may be shown by gradually cooling down a polished surface. When such a surface reaches the temperature that corresponds to the maximum tension of the vapour, a thin film of dew will be deposited, and cloud the surface. If the temperature of the surface be ascertained, we can by means of the above table determine the quantity of vapour contained in each cubic foot. Were the table complete to every degree of the thermometer, simple inspection would give us the weight in grs. opposite to the observed temperature of the surface, provided the air were also of that temperature. But as the temperature of the air is always higher, the vapour will be expanded, and hence, the weight given in the third column, requires to be corrected, by multiplying it by the fractions, representing the ratio of the two expansions, at the temperatures of precipitation, and of the air.

The temperature of precipitation, or that to which the surface is reduced at the moment it begins to be clouded, is called the Dew-Point.

The best instrument that has yet been contrived for observing it, is the Hygrometer of Daniell.

Another instrument, also well fitted for the purpose, has more recently been invented by Pouillet. It is foreign to our purpose to enter into the detail of the structure of these beautiful and ingenious instruments.

CHAPTER X.

Of the Barometer and its applications.

381. The Barometer is constructed by giving to the Torricellian apparatus a support that unites its tube and basin; its form may also be changed in such a manner as to substitute a more convenient receptacle for the mercury than the latter. A scale is then adapted, by means of which the altitude of the column of mercury may be measured in some conventional unit of length.

The French use in their barometers, the metre as the unit, and the scale exhibits its decimal divisions. The English divide their scale into inches, and these are subdivided decimally. The former assume for the mean height of the mercury in the barometer, at the level of the sea, 0.760 metres; the latter, 30 English inches. These heights, however, even when the pressure remains invariable, are affected by the change that takes place in the density of the mercury, under changes of temperature.

When the barometer is intended to remain stationary, in a single place, the original form of a wide basin, in which the end of the tube is immersed, is still used; and as a considerable change in the height of the mercury in the tube will not produce any sensible difference of level in the basin, it has the advantage of needing no correction.

If the open end of the tube be bent upwards, it is called the syphon barometer, and the pressure of the air upon the mercury in the open branch of the tube, will produce the same effect as it does when acting upon that in the basin, in the original apparatus of Torricelli. A scale of the same measure of length must be adapted to each branch of the tube, and the position of the mercury noted upon both, in order to determine the difference of level.

To increase the length of the divisions that correspond to a given change of level in the mercury, various plans have been proposed; all, however, except two, have gone wholly out of use, and therefore require no description. These two are the wheel barometer of Hooke, and the conical barometer.

The form of the wheel barometer, is as follows: adapt a float of iron to the open branch of the syphon barometer, and counterpoise it by a weight attached to a cord passing over a pulley; the weight must be of such magnitude that when the mercury subsides in the tube, the iron float shall preponderate and follow

the mercury in its descent; but when the mercury rises, the float being buoyant upon it, is drawn up by the counterpoise. In this motion the pulley will be turned around, and if an index be affixed to its axis, the latter will traverse around a dial concentric with the pulley. On this dial the divisions may be marked.

The conical barometer is a slender tube with a conical bore, the open end having the largest diameter. It is found that a column of mercury will remain suspended, in such a tube, by the pressure of the atmosphere, if it be carefully inverted, and be not agitated. This column will assume a length which corresponds to the pressure of the atmosphere. If this pressure be lessened, the mercury will fall until the column of mercury, which, as it descends into a wider part of the tube, must decrease in altitude, is again in equilibrio with atmospheric pressure.

If the pressure, on the other hand, increase, the mercury will be forced up; but the length of its column will increase, in consequence of its entering a portion of the tube of less diameter, and the rise will cease when this length becomes the measure of the increased pressure.

It will be obvious that in both these cases, the change in the position of the mercury must be considerably greater than that which will take place in a tube of uniform bore.

382. The invention of the Vernier has, in a great degree, removed the necessity of seeking for a form of the barometer that shall have a scale of greater length than that which corresponds to the change of level in a tube of uniform bore, placed in a vertical position. The scale of the barometer being fixed, the vernier consists in a moveable scale that slides along it, and whose lower or upper extremity carries the index that is made to correspond with the surface of the mercury in the tube. The length of this sliding scale is made exactly equal to some given number of divisions upon the fixed scale; and it is subdivided into a number of equal parts, exceeding, or falling short, by one, the number of divisions upon the corresponding length of the scale.

The theory of this instrument is as follows:

Let a be the length of the fixed scale, which the length assumed for the vernier exceeds or falls short of by 1 division; let n be the number of divisions in a, and which is the same with the number of divisions in the vernier.

The length of the vernier will be,

$$a \mp \frac{a}{n};$$

The length of each division on the fixed scale is $\frac{a}{n}$;

The length of each division of the vernier will be

$$\frac{1}{n}\text{th of } a \mp \frac{a}{n} = \frac{a}{n} \mp \frac{a}{n^2};$$

And as the length of each division of the fixed scale is $\frac{a}{n}$, the difference in the length of the respective divisions is $\mp \frac{a}{n^2}$.

To take the case of the common barometer; the inch is divided into ten parts, and eleven of these are taken for the length of the vernier, which is divided into ten equal parts; a then is equal to 1 inch, $n=10$, and

$$\frac{a}{n^2}=\frac{1}{100}\text{ inch,}$$

which will have the negative sign. Hence the changes in its position will be indicated by looking down its scale, and counting the number of that division, reckoned from the upper end of the vernier, that corresponds with a division of the fixed scale. The index is therefore placed at the top of the vernier; the inch and tenth next below the index, give the height to the first place of decimals, and the second place is given by the indication of the vernier.

In barometers where a greater degree of accuracy is required, the inch is divided into 20 equal parts; the vernier is made equal in length to 24 of these, and is divided into 25 equal parts. In this case,

$$a=1.25\text{ inch,}$$
$$n=\ 25$$
$$\frac{a}{n^2}=\frac{1.25}{625}=0.002.$$

The difference in the length of the respective divisions then is $\frac{1}{500}$ part of an inch, and its sign is positive; hence, the index is placed at the bottom of the vernier, and its indications sought by counting upwards from the index, until that division be reached, which exactly corresponds to a division of the fixed scale.

383. In some of the applications of the barometer, it is necessary that it should be safely portable from place to place. This may be effected in various ways.

(1.) The mercury may be enclosed in a leathern bag, adapted, instead of a basin, to the bottom of the tube. If a screw be applied beneath the bag, the mercury may, by its pressure, be forced up until it strike against the top of the tube; the instrument may then be carried, in an inverted, or in a horizontal position, without risk or danger from the striking of the mercury against the top of the tube.

(2.) The open end of a syphon barometer, may be made of two pieces, united by a stopcock of a material that is not acted

upon by mercury. On inclining the tube, the mercury strikes against its top and fills it. In this position, if the stopcock be closed, the mercury will not be able to escape when the vertical position of the instrument is restored; neither can it oscillate, for it will completely occupy the whole of the tube.

(3.) In the barometer of Gay Lussac, the necessity of a stopcock on the open branch of the syphon, is obviated by contracting the tube at the bend to very small dimensions, and continuing this contraction for some distance up the closed branch of the tube. The external air cannot enter through the tube thus contracted, and it may, therefore, be safely inverted, and carried from place to place.

(4.) In the very perfect and complete portable barometer of Sir George Shuckburgh, the mercury is enclosed in a wooden cistern, closed at the bottom by a flexible diaphragm of leather. This is moved by a screw until the mercury fills the tube. Over the mercury an ivory float is placed, that is brought by the action of the screw to a mark on the stem, that shows when the mercury is at the level whence the divisions on the scale have been measured. This float has a ring between it and the mercury that is pressed up when the mercury rises, and closes the opening through which the float passes.

(5.) In the portable barometer of Englefield, a cistern of wood is cemented to the glass tube, and the whole is packed in a rod of wood, of the size of a common walking cane. A piston works in the cistern, by means of a screw, and can be raised until it force the mercury to the top of the tube. The construction of this instrument has been much improved by Daniell; partly in some particulars that will be stated in the next section, and partly by applying a cistern of cast-iron, and adapting a table to the instrument, by which the correction for the expansion of the glass and mercury, by heat, can be obtained by inspection. The comparative diameters of the tube and cistern are also given, and the height at which the scale has been adapted to the tube. At all other heights, a correction must be applied for the change of level in the cistern, for which this relation between these diameters is the element.

384. In filling the barometer with mercury, and applying the scale, a variety of precautions are necessary. The mercury must be purified by chemical means from all extrinsic substances, for they will alter its density; but when properly purified, the density and character of the mercury are always identical.

The mercury must be completely purged of air, and air must be carefully excluded from the tube; for even a small quantity of air, rising to the top of the tube, will produce a considerable

depression in the column of mercury. To separate both the air that is contained in the mass of mercury, and that which it cannot fail to imbibe in the act of being poured into the tube, the mercury must be boiled in the tube itself. This is done by filling the tube at first only to a third part of its length, and boiling; mercury, that has been heated, is then added in several distinct portions, and each successive portion is heated until it boils. After the tube is nearly filled, as it would endanger it to complete the boiling, the residue is added from a parcel of mercury that has been boiled separately. The reason of adding mercury that has been previously heated, is to prevent the glass from breaking by being suddenly cooled. The effectual exclusion of the air may be ascertained after the tube has been inverted in the cistern, by inclining the tube until the mercury rises to the top; if it strike hard, and with a sharp sound, the air has been completely driven off.

In affixing the scale, it is usual to set it by comparison with another barometer; but Daniell has, in all the barometers made under his direction, applied scales divided by actual measurement from the surface of the mercury in the basin; and the position of the mercury at the time that this division is performed, is noted upon the outside of the case of the instrument, as the neutral point, as has already been stated in the preceding section.

The introduction of air, when the barometer is inverted for carriage, and again restored to its proper position for use, is difficult to be avoided, in barometers of the usual construction; for there is no adhesion between glass and mercury, by which the passage of an extrinsic substance can be prevented. To obviate this defect, Daniell has welded a ring of platinum to the bottom of the tube; between this and the mercury there is an attraction, that will prevent the entrance of air.

Besides the correction for temperature, a correction is required for a depression caused in the mercury by capillary action.

365. Although the mean height of the barometer at the level of the sea is about 30 inches, yet it is far from standing constantly at that height. It is found on the contrary to vary in a greater or less degree at every point on the surface of the earth, and variations of the same character are found to take place at all altitudes that have been reached. These variations are either periodical or accidental. In equatorial regions the former are the more important; but in temperate climates, the periodic variations appear, on a first inspection, to be completely masked by those which are accidental. The whole amount of variation too, appears to increase as we recede from the poles to the equator, although it is influenced in a very great degree by local circumstances. Thus

at New-York the variation does not much exceed 1½ inches, while in Great Britain it is as great as 3 inches.

To separate the accidental from the periodic variations in the barometer, a long series of observations must be made, at hours of the day chosen for their adaptation to the purpose. If the mean height be alone sought, the hour of noon is well suited, and a series of observations, continued for some years, will show whether there be any change due to the season of the year. For the horary oscillations, more frequent observations must be made. In temperate climates, Ramond has proposed as best adapted to the purpose, the hours of 9 A. M., noon, 3, and 9 P. M. Daniell has directed the use of the hours 3, and 9 A. M., and 3 and 9 P. M. If no more than three observations are to be made, the latter author has chosen 8 A. M., 4 P. M., and midnight. Under the equator, Humboldt has shown that the maximum of height takes place at the hours of 9 A. M., and 11 P. M., the minimum at 4 A. M., and 4 P. M. In Paris, Ramond has shown that the times of maximum and minimum, vary with the season; in winter the hours of the maximum are 9 A. M., and 9 P. M., of the minimum at 3 A. M; in summer the maxima occur at 8 A. M., and 11 P. M., the minimum at 4 P. M. These horary variations appear to be less in high latitudes than at the equator, while the accidental variations follow, as has been stated, a different law.

386. These variations are the consequence and indications of changes in the pressure of the atmosphere. Those which we have styled accidental, are from long experience found to produce changes in the state of the weather. The nature of the change portended, is however different in different countries, and there is but one general rule, namely, that a sudden fall of the mercury always portends a high wind.

The barometer may therefore be used to prognosticate the state of the weather, and will be effectual for this purpose, wherever a number of observations has been made sufficient to detect the law, that the variations of the one follow in respect to the other. It is also used to great advantage on ship-board, to foretell gales of wind.

387. As the barometer furnishes a measure of the pressure of the atmosphere, and as this pressure varies, § 366, with a change in the distance from the mean surface of the earth, or with the altitude of the place of observation above the level of the sea, the barometer is used for the purpose of measuring differences in altitude, and ascertaining the absolute height of places above the ocean. The principle on which this may be performed, has already been stated in § 366.

If the atmosphere were of uniform temperature throughout, we have from (381), for the value of the difference of level z,

$$z = m.(\log. P - \log p);$$

or, as the columns of mercury in the barometer are the measures of the pressures, P and p, we may consider those letters as representing the number of inches and decimals in those columns respectively.

The value of the constant co-efficient m, is determined by experiment, and is, by the observations of Gen. Roy, at the temperature of melting ice, and estimated in English measure, 10000 fathoms. Hence,

$$z = 10000 \, (\log. P - \log. p),$$

or in feet,

$$z = 60000 \, (\log. P - \log. p). \tag{393}$$

The height of the columns of mercury is affected by temperature; hence, a correction must be applied to the observed columns of mercury to reduce them to a common temperature.

Mercury expands itself 0.0001016 of its bulk at 32°, for every degree of heat: hence, if the temperature of the columns of mercury be known, the reduction of each to that standard temperature is easy, and the mode obvious. No sensible error can arise, however, in ordinary atmospheric temperatures, from considering the above fraction as the rate of expansion between the temperatures which the mercury in the barometer has at the two stations. Hence, the correction may be applied to but one of the columns, and it is most convenient to do so, to that which has the lowest temperature; and which will most commonly be that observed at the highest station. If T,′ and t', be the two temperatures, the co-efficient denoting this correction will be

$$1 + 0.0001016 \, (T' - t'). \tag{394}$$

Difference of temperature will also affect the density of the air; and hence the difference of the height of the mercurial columns will not be the same at other temperatures, at each or either of the places, as it would be, had both the temperature of 32°. A correction is therefore needed for this cause, which will affect the co-efficient m. Air dilates, as has been stated, § 367, 0.002083 of its bulk, for every degree of temperature reckoned from 32°. This fraction, therefore, is not constant at all temperatures. It is, however, usual to consider it as such, and to apply it, by means of the mean temperature of the air at the two stations, above the freezing point: hence, the correction consists in multiplying m by

$$1 + 0.002083 \left(\frac{T + t}{2} - 32° \right),$$

or which is the same, by

$$1 + 0.0010415 \, (T + t - 64°). \tag{395}$$

The formula (393) therefore becomes, when these corrections are taken into account,

$$z = 10000\,[1 + 0.001041(T + t - 64)]\,,\quad \log.\frac{P}{p\,[1 + 0.0001016\,(T' - t')]}\,. \qquad (396)$$

This formula is sufficiently near the truth for most of the cases that can occur in practice. When much accuracy is required, and especially where the difference of level is great, there are other circumstances to be taken into account.

(1). The mean height of the mercury in the barometer is affected by the difference in the apparent force of gravity at different latitudes, according to the law in § 100. A correction, therefore, may be needed on this account; the element of which is the latitude of the place. This, however, is at most small, and is generally neglected.

(2). A more important cause of error exists in the mixture of atmospheric air with aqueous vapour, which is subject to different laws of pressure and equilibrium. It has been considered by an authority of no less weight than that of Laplace, to be impracticable in the present state of our knowledge, to apply a correction for this circumstance. This difficulty has, however, been overcome by Daniell, who has shown that his hygrometer can be applied to the purpose; and has published tables that accompany it, by means of which this correction can be applied. These tables, and the principles on which they are founded, may be seen on reference to the "Quarterly Journal of Science, edited at the British Institution", No 25.

388. In applying the barometer as a measure of differences of level, several precautions are necessary. In order to prevent the observations being affected by the periodic, and still more by the accidental variations in the mean pressure of the atmosphere, they should be made simultaneously at the two places whose difference of level is sought. Hence, two observers, and two instruments, are necessary.

The barometers should each, have a thermometer inclosed in their case, in order to mark the temperature of the mercury they contain. These are called the Attached Thermometers.

Each observer should be furnished with a separate thermometer, to note the temperature of the air.

One of the observers remaining stationary, the other may move with his instruments from place to place, and thus observe at a number of stations. In order to a comparison of observations, the observer who remains at the same place should take observations at prescribed intervals of time, say from 10′ to 15′; the one who proceeds from place to place, should note the time of each

of his observations. Those nearest in point of time, must of course be taken as the objects of comparison.

The calculation is performed as follows: The column of mercury at the station whose temperature is lowest, is to be corrected for that element. This correction consists in adding to the height of that column, its continual product by the constant fraction 0.0001016, and by the difference of the indications of the two attached thermometers.

The difference between the logarithms of the other column of mercury, and this corrected column is then to be taken.

This difference multiplied by 10000, is the approximate difference of altitude between the two stations in fathoms, and is obtained at once, by moving the decimal point four places to the right.

The approximate difference of level is lastly to be corrected, for the difference of the temperature of the air at the two stations from 32°. This correction is obtained by multiplying the excess of the mean of the two detached thermometers above 32°, or their defect below 32°, by the constant fraction 0.002083, and the product by the approximate altitude. This correction is added when this mean temperature exceeds 32°, and subtracted when it is less.

389. If the correction for the moisture of the atmosphere, whose element is obtained from the indications of the hygrometer of Daniell, is not employed, it will be better to substitute for the constant fraction 0.002083, which represents the expansion of dry atmospheric air for each degree of Fahrenheit's thermometer, the fraction 0.00244, which by the experiments of Gen. Roy, is consistent with a mean state of moisture.

The co-efficient m, which we have stated at 10000 fathoms, has been inferred by Raymond, from a great number of observations, to be 18336 metres, at the latitude of 45°. This is equivalent to 10025 fathoms, and makes the number given in our formula in error, about $\frac{1}{400}$th part. The whole change in the intensity of gravity from the pole to the equator is, as has been shown, § 100, $\frac{1}{300}$th part of the force of gravity, on the hypothesis of the earth's having a spherical figure; but in consequence of the spheroidal figure of the earth, the apparent intensity is still farther lessened at the equator, and the ratio of the two forces is, § 290, actually as great as $\frac{1}{176}$, or more exactly 0.005674.

As the value of m is determined for the latitude of 45°, it becomes necessary to reduce observations at other latitudes, to the latitude of 45°. The value of the correction may be thus found:

Let g be the force of gravity at the equator; f the centrifugal force there, will be $=0.005674\ g$.

From (294), we have for the value of γ, the force of gravity in lat. 45°,

$$\gamma = g + f \sin.^2 45° = g + \tfrac{1}{2} f;$$

whence

$$g = \gamma - \tfrac{1}{2} f;$$

for the force g', at any other latitude L, we have

$$g' = g + f \sin.^2 \mathrm{L};$$

substituting the value of g, from the foregoing equation,

$$g' = \gamma - \tfrac{1}{2} f + f \sin.^2 \mathrm{L},$$

or

$$g' = \gamma - \tfrac{1}{2} f (1 - 2 \sin.^2 \mathrm{L});$$

but as

$$\cos.\ 2\,\mathrm{L} = 1 - 2 \sin.^2 \mathrm{L};$$

we obtain

$$g' = \gamma - \tfrac{1}{2} f . \cos.\ 2\,\mathrm{L};$$

and substituting the value of f,

$$g' = \gamma - 0.002837 \cos.\ 2\,\mathrm{L}.$$

Whence a correction may be deduced to be applied to the quantity z, in the formula, when the height is considerable, and the latitude distant from 45°.

The formula for the complete calculation then becomes,

$$z = 10000\,(1 - 0.002837 \cos.\ 2\,\mathrm{L}) . \left[1 + 0.002083 \left(\frac{\mathrm{T}+t}{2} - 32\right)\right] . \log. \frac{\mathrm{P}}{p[1 + 0.0001061\,(\mathrm{T}' - t')]} . \qquad (397)$$

We annex a form of the calculation

EXAMPLE.

Calculation of the difference in level of two stations, A and B, at which the following observations were made cotemporaneously.

Stations.	Barometers.	Detached Thermometers.	Attached Thermometers.	Lat.
A	28.691	$\mathrm{T} = 84°$	$\mathrm{T}' = 76°.5$	45°.32′
B	28.791	$t = 78°$	$t' = 82°$	46.

$$\frac{\mathrm{T}+t}{2} - 32° = 49° \quad \mathrm{T}' - t' = -5°.5 \quad 2\mathrm{L} = 91°.32'$$

log. 0. 002837=	7.45286	log. 28.691=	1.45775
log. cos. 91°.32′=	—8.42746	log. 27.791=	1.44390
log.—0.000076=	5.88032		

log. [1+(0.0001061 × —5.5)=0.99946]= 9.99975

1.44365

141=10000×0.01410

log.	- - - - - - 141	=2.14922
log.	[1+(0.002083 × 49)=1.10207]	=0,04221
log.	- [(1—0.000076)=0.999924]	=9.99999
log.	- - - - 155.39 fathoms	=2.19142
	6	
	932.34 feet.	

If we take m=60150 feet, the last part of the calculation will be,

log.	0.0140	=8.14922
log.	60150	=4.77924
log.	1.10207	=0.04221
log.	0.999924	=9.99999
log.	934.68 ft.	=2.97066

CHAPTER XI.

Of the Attraction of Cohesion.

390. The conditions of the equilibrium of fluids that have been investigated in the preceding chapters, are occasionally affected by an action that takes place between their particles, and those of solid bodies in contact with them. This action is called the Attraction of Cohesion, or from the most remarkable class of the phenomena to which it gives rise, Capillary Attraction.

The existence of this attraction, and its capability of exerting a determinate force, may be shown by a very simple experiment.

If a disk of any substance that is capable of being moistened by a liquid, be suspended from the arm of a balance, and counterpoised by weights; and if a vessel containing the liquid be raised from beneath, until its surface just touch the disk, they will be found to adhere. This adhesion may be overcome by adding weights to the opposite arm of the balance; or rather the disk may be drawn away from the mass of liquid, for it will still carry with it a film of the liquid; and the force exerted by the additional weight does not overcome the cohesive force, but only the attraction of aggregation that exists between the particles of the liquid.

391. Phenomena due to the same cause are observed in a variety of cases. Thus: the surface of water or alcohol, in a glass vessel, is slightly raised around the edges; if the glass be diminished in size, the elevation of the fluid at the edge will increase; and in a tube of glass of small diameter, a column will be supported within it, when plunged in a liquid, above the level of the general surface. In tubes of very small diameter, called, from their size, Capillary Tubes, this column may amount to some inches.

In other cases, a depression exists within the tube; thus, when one of glass is immersed in mercury, that liquid will be obviously lower within the tube than it is without.

In cases where the liquid is raised in a tube, its upper surface assumes a concave form, which, in small tubes, differs but little from a portion of a sphere; and in cases where a liquid is depressed, the upper surface is convex.

Similar phenomena occur between two tubes of different diameters, placed one within the other, and between two parallel plates. If two plates are inclined to each other, and meet at one

of their edges, the liquid will rise between them, its surface assuming the form of a curve.

Two solid bodies that would not otherwise adhere, may be made to stick together with considerable force, by this action. Thus, if two plates of glass be moistened with water, and then pressed together, they require a considerable effort to separate them; and two plates of polished brass may be in the same manner united by oil or melted tallow. In the latter case, the interposed substance becomes solid on cooling, and the force with which it resists an effort to separate the plates, becomes very great. In general terms, when two solid bodies are made to cohere by the intervention of a substance that can be applied in a liquid state, and which afterwards becomes solid, the cohesion is rendered more intense. This principle is applied in the process of soldering the metals.

392. In order to examine the phenomena of the rise of liquids in capillary tubes:

Let us suppose a prismatic tube, standing in a vertical position; that its sides are perpendicular to its base; and that it is supported in a vessel, in such a manner that it plunges at its base into a liquid of such a nature as to rise in the tube above its natural level.

The attraction of the tube has a very limited sphere of action, for the height to which fluids rise in tubes of different thickness, provided their interior diameters be the same, is constant. Any small column of the liquid, situated in or near the axis of the tube, will not be affected by this attraction, but must be supported by the action of the adjacent columns of fluid. It is therefore clear, that the action of the tube upon the column immediately in contact with it, is the final cause of the elevation of the whole mass. A ring of the fluid is first raised; this raises a second; the second a third, and so on, until the weight of the fluid exactly balances the attractive forces that are exerted by the sides of the tube. In order to determine the conditions of equilibrium, let us conceive the tube to be produced in the form of a syphon, by a part of no thickness, and which, therefore, does not by its attraction interfere with the conditions of equilibrium, and does not prevent the re-action of the fluid particles contained within it upon those in the tube. It is obvious that the column contained within this imaginary tube, will exactly replace, in its fluid action, the whole mass contained in the vessel; and in the case of equilibrium, the pressure in the two branches of the tube must be identical. But the columns of fluid in the two branches are of unequal heights; the difference of pressure that results from this inequality, must, therefore, be counteracted by the attractions of the prism and the fluid; these are exerted in a vertical direction in the original branch of the tube.

As the prism is assumed to be vertical, its base is horizontal. The fluid contained in the additional tube, is attracted vertically downwards: (1,) by its own particles; (2,) by the fluid that surrounds it. But these two attractions are counteracted by the like attractions that the fluid contained in the second vertical branch sustains. The fluid of the vertical branch of the second tube is besides attracted vertically upwards by the fluid in the first tube. But this attraction is destroyed by the attraction the former exerts upon the latter column of fluid. These reciprocal attractions may therefore be disregarded.

Finally the fluid in the second tube is attracted vertically upwards by the first tube; hence there results a vertical force that contributes to destroy the excess of pressure due to the elevation of the fluid in the first tube. This force tends to destroy the excess of pressure exerted in opposition to it, by the column of fluid raised in the original tube above the general level.

This force we shall call P.

The forces that act upon the fluid contained in the original tube, are as follows:

(1.) It is attracted by itself; but as the reciprocal attractions of the particles of a solid body do not cause any motion, we may abstract this attraction, for we may, in a tube of uniform bore, standing in a vertical position, consider the vertical pressure as if it were produced by a solid body filling the tube.

(2.) The elevated fluid is attracted downwards by the liquid column contained in the part beneath the level of the external fluid; but it attracts in its turn with an equal force, and these attractions mutually destroy each other.

(3.) The fluid is also attracted downwards by the fluid that surrounds the ideal prolongation of the tube; hence there results a vertical force directed downwards, that we shall call $-P'$, its sign being negative, in order to represent that its direction is opposite to that of the force P. The forces would be exactly equal if the tube were composed of the same material with the fluid. Their inequality is therefore due to the difference in the intensity of the attractive forces exerted by the particles of the fluid upon each other, and by the particles of the tube upon those of the fluid. If we take r and r' to represent these respective intensities, we have

$$P : P' :: r : r'. \tag{398}$$

(4.) The fluid in the first tube is attracted vertically upwards by the matter of the tube, and this force is obviously equal to P.

The whole of the attractive forces that act, are, therefore,

$$2P - P', \tag{399}$$

and they are equal to the weight of the column that they raise.

If g represent the force of gravity, D the density of the fluid, and V the volume of the elevated column,

$$g\,DV = 2P - P'. \tag{400}$$

It is obvious from this equation, that the quantity V, will always have the same sign with the quantity 2 P—P′.

Hence:

393. When the attraction of the particles of the fluid for each other is half that which the particles of the tube have for those of the fluid, the level within and without the tube will be the same; when the former is less than half the latter, the fluid will be raised; but when it is greater, the surface of the fluid will be depressed.

The action being exerted at imperceptible distances, will only be felt by the columns in immediate contact with the tube; we may, in consequence, neglect the curvature of the tube, and consider it as developed into a plane surface.

The attractive force, P, will therefore be proportioned to the size of this plane, and may be represented by r C, C being the contour of the surface. And for a like reason,

$$P' = r' C;$$

whence

$$g DV = (2r - r') C. \qquad (401)$$

This formula is applicable to all the cases that have been observed in practice, and the results that flow from it are consistent with observation.

In cylindrical tubes:

Let a be the radius of the interior of the tube, h the height of the column, measured from the level of the fluid without, to the curved surface it supports or depresses. The volume of this column is

$$\pi a^2 h. \qquad (402)$$

To this, must be added the volume of the meniscus in which it terminates, which will be the difference between the volume of a cylinder, whose height and base are both equal to a, and the hemisphere whose radius is a, or to

$$\pi a^3 - \frac{2\pi a^3}{3}, \text{ or simply to } \frac{\pi a^3}{3}.$$

The whole volume of fluid raised, will therefore be

$$\pi a^2 h + \frac{1}{3} \pi a^3;$$

and the contour of the base C, is

$$2\pi a;$$

substituting these values in the foregoing equation, we have

$$g D (\pi a^2 h + \pi a^3) = (2r - r') 2\pi a; \qquad (403)$$

and dividing by πa,

$$a\left(h + \frac{a}{3}\right) = 2\left(\frac{2r - r'}{gd}\right). \qquad (404)$$

If the tubes be of the same material, and be plunged in the same fluid, whose temperature is constant, r, r', g and D will be the same in every case, and the second member of the equation will be a constant quantity, which we may call A, and

$$a\left(h+\frac{a}{3}\right)=A. \tag{405}$$

whence

$$h+\frac{a}{3}=\frac{A}{a}. \tag{406}$$

When the tube is small, the height, h, is great, compared with the radius a, and the quantity $\frac{a}{3}$, may be neglected, or will be masked by the errors of observation; we may therefore assume,

$$h=\frac{A}{a}. \tag{407}$$

Hence:

394. In capillary tubes, liquids will rise or be depressed, according to the relation between the attractive forces stated in the preceding section, to heights inversely proportioned to the diameters of the tubes.

Applying the formula (402), by a similar method, to the case of two parallel solid surfaces, at a small distance from each other, which we shall call d, we would obtain

$$h=\frac{A}{d}. \tag{408}$$

Hence:

A liquid will be raised or depressed between two parallel plates, at small distances from each other, to heights that are inversely proportioned to the distances between the plates; and these heights will be the same as those to which the same liquid would rise in a tube of the same material with the plates, whose radius is equal to the distance between the plates.

Two concentric tubes, or a tube surrounding a solid cylinder, may be considered as a case of this kind; the liquid will rise between them to half the height it would in a tube whose diameter is equal to their distance.

Such is the theory of Laplace, which, neglecting certain small quantities, coincides with the observations of former experimenters, but which by the more accurate experiments of Gay Lussac, has been found to be true, even in its full extent; the refinements introduced by the latter, having enabled him to detect the small variations in level that had escaped those who preceded him.

Laplace has also investigated the nature of the surface that

would be assumed by a liquid. We have not space, nor would it be consistent with an elementary treatise, to enter into his beautiful and complete analysis; we shall, therefore, content ourselves with stating the results of observation which are confirmed by the theory.

In tubes of small diameter, the surface of the liquid, whether elevated or depressed, is always spherical; but in the former case, it is a concave portion, in the latter, a convex one, of a sphere.

395. Aëriform fluids are also affected by capillary attraction; and this is, in some cases so intense, as to condense them into a volume considerably less than that which they occupy under ordinary pressures. The most remarkable instance of this sort is, the absorption of the gases by charcoal. This substance, when recently burnt, takes up within twenty-four hours, according to the experiments of Saussure, of

Ammonia . .	90	times its	volume.
Muriatic Acid Gas	85	"	"
Sulphurous Acid .	65	"	"
Sulphuretted Hydrogen	55	"	"
Nitrous Oxide .	40	"	"
Carbonic Acid .	35	"	"
Olefiant Gas .	35	"	"
Oxygen . .	9.25	"	"
Nitrogen . .	7.5	"	"
Hydrogen . .	1.75	"	"

Solutions also, generally ranked among chemical phenomena, are unquestionably due to the mechanical action of the particles of the solvent upon those of the solid; and this attraction is opposed by the attraction of aggregation that the particles of the latter have for each other. The mixture of liquids, particularly when attended with the diminution of bulk called Concentration, may be also included among the cases of cohesive attraction.

BOOK VI.

OF THE MOTION OF FLUIDS.

CHAPTER I.

Theory of the Motion of Liquids.

396. We may, by the application of the principle of D'Alembert to the equations of the equilibrium of fluids, deduce the general equations of their motion. These equations are, however, extremely complicated, and are incapable of complete integration. It has, therefore, been more usual to proceed by means of an hypothesis, that appears at first sight to approximate to the truth. This hypothesis considers the fluid to be divided into a number of horizontal layers or stràta, and that the particles in each of these separate layers have a common velocity. Each of the layers then would continue parallel to itself, and is composed of the self-same particles throughout the whole duration of its motion; and any given layer will descend and occupy the place of that which is immediately beneath it, and so in succession.

If this hypothesis be applied to the case of a liquid contained in a vessel of irregular figure, it might be demonstrated, that in moving through it, each different layer will have a velocity inversely proportioned to the area of the section of the vessel at which this layer is situated.

This hypothesis is, however, in many cases, at utter variance with what is observed in practice. When, for instance, a liquid is placed in a prismatic vessel that is permitted to discharge itself through an orifice in the bottom, the surface ceases at once to be level, being depressed immediately above the orifice. The surface, therefore, becomes concave, and the particles that compose it, instead of tending to descend in a horizontal layer, appear to move, as it were, upon an inclined plane, to the point that is most depressed, and thence to descend vertically to the orifice. Not only does this tendency to the vertical line appear

at the surface, but it is also manifested in the inferior strata: thus, if a liquid of less density be poured upon the surface of the liquid that is first introduced into the vessel, it speedily joins the descending current and passes out; but it does not pass out unmixed, for the two liquids are intimately mingled, until the whole of the lighter be discharged. This tendency in the lower strata, towards the column that is immediately above the orifice, is also manifested by placing in the vessel powders of a density equal to that of the liquid; these will be seen to move towards the orifice, in curves of various degrees of convergence, and will unite themselves with the effluent stream.

397. These phenomena may be accounted for, by supposing that each particle of the liquid descends towards the orifice exactly as if it were unconnected with the surrounding mass. It would therefore acquire an uniformly accelerated velocity; the column would be broken, and spaces left between the particles, towards which the pressure of the adjacent liquid would impel other particles, that would thus join in the current, and occupy the void spaces of the column.

If the orifice be pierced in the side of the vessel, the particles of liquid will still move towards it like falling bodies, but will describe a curve instead of a vertical line. If the orifice be made in a part of the vessel that will permit it to be directed upwards, the particles will again reach it, under circumstances similar to those which are found in bodies moving, under the action of gravity, upon curved surfaces whose tangents make with a horizontal line, the same angle that the direction of the orifice makes with the horizontal plane.

When the motion begins, the particles immediately in contact with the orifice, move from a state of rest; and those that lie between them and the surface cannot be accelerated without accelerating those beneath them; thus a resistance will be opposed to the descent that will for a time prevent the particles that proceed from the surface of the liquid, from acquiring the velocity a falling solid body would attain in passing through the same space. The time for which this resistance will produce an appreciable effect is but short, for so soon as the first particle that issues shall have fallen through a space equal to the altitude of the fluid above the orifice, its velocity will become equal to that the particles proceeding from the surface, would acquire at the orifice; and it will no longer retard the column that follows it.

If the fluid move in a vessel of variable section, its velocity does not vary with the area, but there is a column or vein that moves in it, precisely as if it were in a prismatic vessel, while in the parts whose areas are greatest, eddies are formed. The nature

and character of these eddies will be more particularly considered hereafter.

The vein that moves according to the law of gravity, will be resisted by the viscidity of the neighbouring particles of the liquid; it will also be retarded, in consequence of the particles that join it in its course having a less velocity than those which proceed from a higher level. The former will receive a part of the motion of the latter, and the whole will move forward with a common velocity. If the vein nearly fill up the vessel in which it is moving, it will meet with a resistance analagous to friction from the sides, and will be influenced by the attraction of cohesion.

If these circumstances be left out of account, we may consider the velocity of any part of a vein of a gravitating liquid in motion, to be such as would be due to the height of the surface of the liquid, above the point whose motion is considered. In some cases, the circumstances to which we have referred are of no moment, and may safely be neglected; in others, they affect the velocity in a high degree, and even render constant that motion which would otherwise be uniformly accelerated.

It therefore becomes necessary to distinguish different cases of the motion of liquids. The more important of these cases are five in number: viz.,

(1.) The motion of liquids that issue from orifices pierced in thin plates.

(2.) The motion of liquids through orifices cut in thick plates, or through short tubes adapted to orifices.

(3.) The motion of liquids in long tubes or pipes.

(4.) The motion of liquids in open channels.

(5.) The motion of liquids over the edges of the lower portion of the irregular sides of the vessel or basin that contains them.

We shall take up the consideration of these several cases in the order in which they have been mentioned.

CHAPTER II.

OF THE MOTION OF LIQUIDS THROUGH ORIFICES PIERCED IN THIN PLATES.

398. If we abstract the circumstances of which we have spoken in the close of the last chapter, a liquid, on reaching an orifice, will have a velocity due to the height of the level of the liquid above the orifice. To represent this in a formula:

Let v be the velocity with which the liquid issues; h the vertical height of the surface of the liquid above the orifice; g the measure of the force of gravity; then by (61),

$$v=\sqrt{2gh}. \tag{409}$$

From this formula a variety of consequences immediately follow.

(1.) If the vessel be kept constantly full, the velocities of the effluent fluids, from any orifice given in position, is constant.

(2.) From orifices pierced at different heights in the side of a vessel kept constantly full, the velocities are as the square roots of the depths of the orifices beneath the level surface of the liquid.

(3.) If the vessel be permitted to empty itself, the velocity with which the liquid will issue from a given aperture is equally retarded.

(4.) As none of the circumstances of which we have spoken will affect the area of the column of liquid, discharged from a given orifice, the quantities discharged in the elements of the time will be directly proportioned to the velocities.

Therefore, as in a vessel permitted to empty itself, the velocities are uniformly retarded, it is obvious from § 49, that twice as much liquid should flow from a given orifice, in the unit of time, when the vessel is kept constantly full, as should flow from the same orifice in the same time, when the vessel is permitted to empty itself; and so, when the vessel is permitted to empty itself, it should occupy twice as much time to discharge a given quantity, as would suffice for an equal discharge from a vessel kept constantly full.

(5.) In emptying a vessel through an orifice in its bottom, the quantities discharged in equal times should decrease as the series of odd numbers.

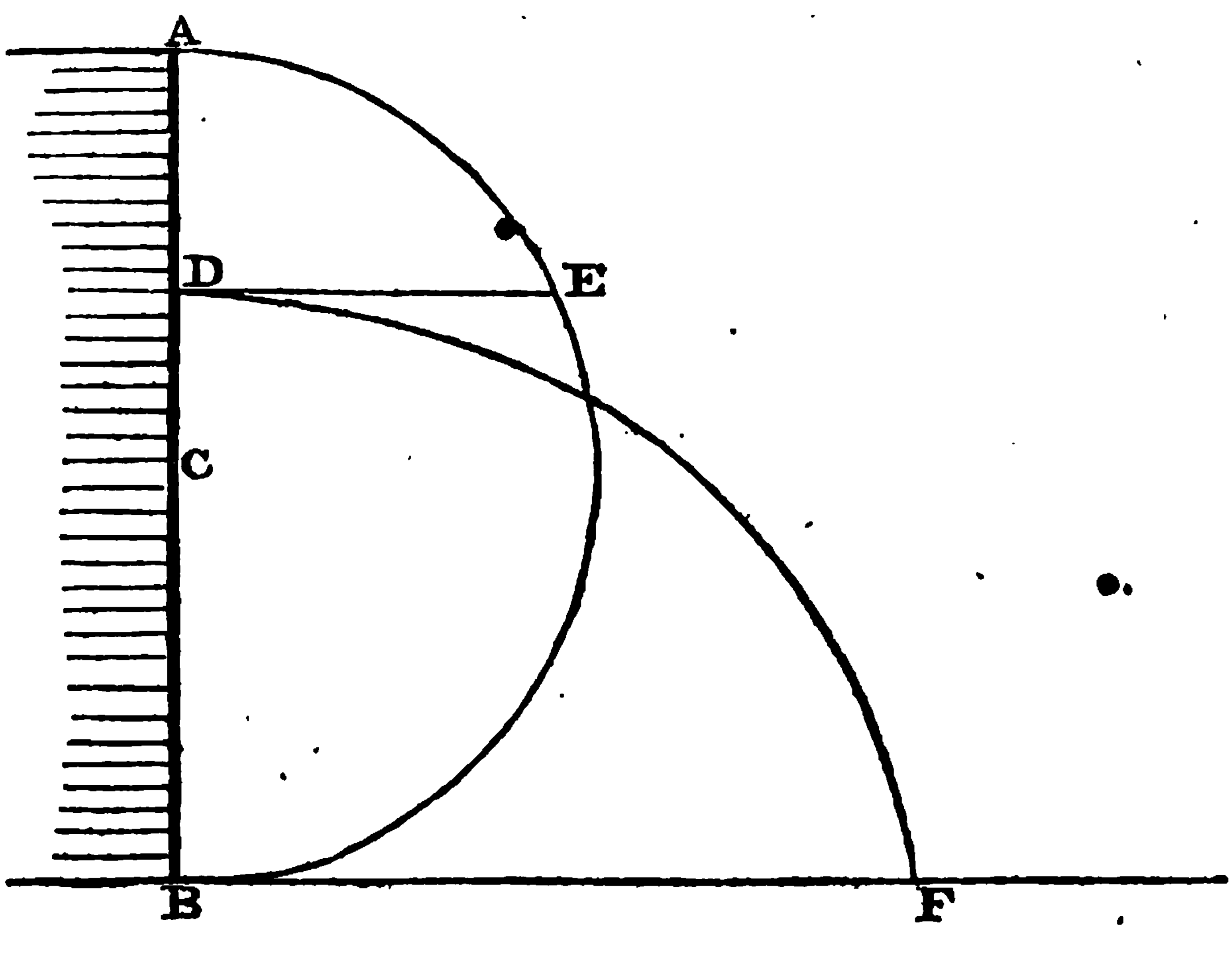
(6.) When a fluid issues from a vessel in a vertical direction, it will rise or fall in a vertical line; and if it be directed upwards, as it has an initial velocity due to the height of the surface of the liquid above the orifice, it should rise, if we abstract from the resistances it meets, to the level of that surface. The velocity in the jet will be uniformly retarded.

(7.) If a liquid spout from an orifice in any other than a vertical direction, the joint action of its effluent motion, with a velocity due to the depth, and of the force of gravity, would, if no other forces acted, cause it to describe a parabola, whose directrix will lie in the horizontal plane, coinciding with the surface of the liquid, (§53).

(8.) If different orifices be pierced in a horizontal direction, in the vertical sides of a vessel, which is kept constantly filled with a liquid, the effluent velocities of each vein of liquid will vary, and each will have a different distance to descend before it reaches the horizontal plane that coincides with the bottom of the vessel. Hence the curves described in each case will be different.

The comparative distances to which the several jets will pass over this horizontal plane may be thus investigated:

Let AB be the side of a prismatic vessel, at the point D, in which an orifice is pierced, whence a fluid kept constantly at the level of the point A issues in a horizontal direction. Bisect AB in C, and around C describe a semicircle; draw the ordinate DE through the point D.



Let
$$AD=h,$$
$$DB=a,$$
$$DE=x.$$

The liquid, in issuing from the point *d*, will have a velocity due to the height *h*, and equal to

$$\sqrt{2gh};$$

this velocity would carry it with uniform velocity through twice the distance *h*, in the same space of time that it has taken to acquire that velocity. But so soon as it leaves the orifice, it has its direction changed and describes a parabola. Under the action of this deflecting force, it will reach the horizontal plane BF, in the same time that it would have fallen through the height DB, if it were influenced by the force of gravity alone. The times of describing AD, and DB, are respectively

$$\sqrt{\frac{2h}{g}}, \text{ and } \sqrt{\frac{2a}{g}};$$

in the first of these times, the projectile force would carry it with uniform motion through the space $2h$, and in the second, through the distance at which the jet of fluid strikes the horizontal plane, which we shall call *d*. As the spaces are as the times, we have the analogy

$$\sqrt{\frac{2h}{g}} : \sqrt{\frac{2a}{g}} : : 2h : d,$$

whence we obtain for the value of *d*,

$$d=\frac{2h\sqrt{a}}{\sqrt{h}}=2\sqrt{ah}=2x.$$

The distance then from the perpendicular side of the vessel at which the parabolic jet strikes the horizontal plane on which the vessel stands, is equal to twice the line ordinately applied to a semicircle, of which the vertical depth of the fluid is the diameter. From the centre of the circle, or half the height of the liquid, the horizontal range will be the greatest; and at equal distances above or below this point, the horizontal ranges are equal.

399. When a liquid issues from an orifice with a given velocity, whether it be such as is determined from the foregoing abstract theory, or modified by physical circumstances, it might be at first sight concluded that the quantity discharged in the unit of time, might be determined by multiplying the area of the orifice by the velocity. This, however, is not the case. The liquid does not issue in a prismatic shape, and hence its quantity is not measured by the contents of a prism of which the orifice is the base. On the contrary, the vein of liquid is obviously contracted

soon after it issues forth, and again spreads out to dimensions larger than those of the orifice. That this ought to be the case, will be understood from reference to the foregoing theory. For, as the particles move from the lower layers of the liquid to join the vein directed towards the orifice, they have a motion oblique to that of the general current; but from the particles that compose it, they receive a change of direction, and at the same time re-act upon them. Thus the particles that issue from the edges of the orifice, will have a direction inclined to the axis of the jet, and the stream, of which they form a part, must contract in dimensions. As these directions will cross each other, some of the particles will, below the point to which they converge, be forced outward from the axis. Thus the shape of the jet will be one formed of two truncated conoidal frusta; one of these will have the orifice for its base, and a definite altitude; while the other will have the smaller base of the former for its lesser base, and an unlimited altitude. The investigation by analytic means of the exact figure of these conoidal frusta, were it practicable, would yet be attended with no valuable results; for the contraction in the vein is connected with the change in velocity growing out of the viscidity of the liquid, and the mutual action of its particles; therefore, the separate effects of these two different actions cannot be distinguished in experiment. We, in consequence, consider that the true velocity is given by the principles of the preceding section, and that the whole diminution in the quantity discharged, is due to the contraction of the vein.

The contraction that occurs in the form of a jet of fluid, issuing from an orifice, is too apparent to have escaped notice, even at an early period. Newton, however, was the first who attempted to ascertain the amount of this contraction. In this he was not perfectly successful, but conceived that he had found it to be in the ratio 5 : 7, or of $\sqrt{1} : \sqrt{2}$.

If a be the area of the orifice, q the quantity discharged, and we use the same notation as before,

$$v=\sqrt{2gh};$$

and upon the hypothesis that the quantity is found by multiplying the area of the orifice by the velocity,

$$q=a\sqrt{2gh}. \tag{410}$$

This is called the Theoretic Discharge. If it be reduced in the ratio given by Newton, we have

$$q=a\sqrt{gh}. \tag{411}$$

As the change in this formula is made in the second part of it which represents the velocity, a false inference has been drawn

by some writers, who, forgetting the circumstance of the contraction of the vein, have stated that the velocity itself is diminished, and becomes no more than is due to half the height of the fluid above the orifice. This, however, is an obvious error; the velocity is but little affected when the liquid issues from an orifice pierced in a thin plate, and the diminution in the actual discharge compared with the theoretic, is principally due to the contraction of the vein.

The best experiments on the phenomena of liquids issuing from orifices pierced in thin plates, are those of Bossut. From these it appears,

(1). That, except when a vessel is nearly exhausted, no sensible error can arise from considering the velocities as due to the height of the level surface of the liquid above the orifice.

The narrowest part of a jet of liquid, issuing from an orifice, is called the *Vena Contracta;* this is situated at a distance from the orifice, when of a circular figure, that is equal to its radius.

(2). The figure of a vertical jet, lying between a circular orifice and the vena contracta, is nearly a conic frustum, whose two bases have to each other the ratios of 62 : 100, or nearly as 5 : 8, instead of 5 : 7, as stated by Newton.

The figure of the jet, beyond the vena contracta, is also sensibly a conic frustum, the angle of whose vertex is 32°.

In orifices of any other figure, the same ratio is nearly true between the dimensions of the orifice and the section of the liquid at the vena contracta; but the figure of the jet is not pyramidical; at a small distance from the orifice, if of a rectangular shape, it assumes the form of a cross, whose arms lie in the direction of the diagonals of the orifice; beyond this, the section again becomes rectangular, with diagonals parallel to the sides of the orifice, and this figure is retained in the subsequent enlargement of the vein.

Analogous changes of figure take place in the section of the vein of liquid, when the orifice has the figure of a triangle or a polygon.

(3). So long as the vessel continues to hold a column of liquid at a considerable height above the orifice, the actual discharges from orifices of any figure whatsoever, are to the theoretic *nearly* in the ratio $\frac{62}{100}$; and the velocities and quantities are *nearly* proportioned to the square roots of the depth of the liquid.

(4). Small orifices discharge rather less than the reduced quantity, large orifices rather more; and of orifices of equal area and unequal circumferences, those with the smallest circumference discharge the greatest quantity.

400. When the liquid spouts vertically upwards, there is a deviation from the theoretic height that can be easily perceived. That this should be the case, will be obvious when we consider that the particles of liquids are retarded by the column that has preceded them, the particles of which are moving with a continually diminishing velocity; and that the particles, after reaching their utmost height, tend to return in a vertical direction: a part of the force of the ascending column must therefore be applied to force them to one side. In consequence of the latter circumstance, it has been found that a jet of liquid, when slightly inclined, rises higher than if pointed vertically upwards. When the original velocity is due to a great height, and is, in consequence, large, the resistance of the air becomes a powerful retarding force, and hence creates a limit beyond which no head of water, however great, can cause a vertical jet of liquid to rise. The height to which a liquid rises vertically upwards, and which is, therefore, always lower than the level of the liquid in the vessel whence it issues, is called the Effective Head.

It has been ascertained by the experiments of Mariotte, that a head of five French feet and an inch, produces a vertical jet of five feet.

If H, and H′, be the actual heads of two masses of water that form vertical jets; h, and h', the effective heads, the experiments of Mariotte, give the following relation between them.

$$\frac{H-h}{H'-h'}=\frac{h^2}{h'^2}. \tag{412}$$

and

$$H=(H'-h')\frac{h^2}{h'^2}+h, \tag{413}$$

or taking the above data, where

$$H'=5\frac{1}{12},$$
$$h'=5$$
$$H=\frac{h^2}{360}+h; \tag{414}$$

whence the real head that will produce a jet of any given height, can be estimated.

401. If the velocity due to the effective head be employed in the parabolic theory, instead of the actual velocity, the results will be nearly identical with those that actually take place. Hence, from the theory of projectiles, § 250, we have for the height to which an inclined jet will rise,

$$h \sin. 2i; \tag{415}$$

and for the horizontal distance to which it will reach,

$$2h \sin. 2i. \qquad (416)$$

402. The rules of § 399, are only found to hold good in practice when the height of the column of liquid in the vessel is large, compared with the area of the orifice; as the height lessens and the vessel becomes nearly empty, the velocity of discharge is diminished below that due to the height; and the contraction of the vein increases. This grows partly out of a rotary motion that often takes place in the fluid in the vessel, and causes a centrifugal force that lessens the action of gravitation. The particles, therefore, that reach the orifice in a vertical direction, have a less velocity than they would otherwise acquire; and those whose direction is oblique, are less powerfully acted upon, and continue their oblique course longer.

The formation of the vortex, whose rotary motion produces these effects, may be thus explained:

The particles that enter the vein directed towards the orifice, have a motion that may be resolved into two, one in the vertical, the other in a horizontal direction. The latter is obviously a centripetal force. If a third force act upon any one of the particles, in any other direction than that of these two components, it will cause the particle to deviate in a horizontal direction; for one of its components will be horizontal, and thus the motion in the direction of the radius will become circular, if the disturbing force be of sufficient intensity, or spiral, if it be less intense. In the latter case, it may be considered as taking place in circles, successively decreasing in magnitude, and the laws of circular motion of § 64, will be applicable. This disturbing force may proceed from extrinsic causes, but it may arise also from irregularity in the figure of the vessel, or from the position of the orifice being in any other point than the centre of magnitude of a base of regular figure; for in either of these cases, the particles that join the vein at a given level, will reach it with different inclinations and velocities, and will therefore effect each other's motions.

If we call the velocity of rotation of any particle v, its distance from the axis of the vein, r, and the time of a revolution, t, the velocity of rotation will be constant; and will, therefore, in the successively decreasing circles, be inversely proportioned to the radii; and the areas of the circles will be as the times of describing them, or the times will be directly as the squares of the radii. The centrifugal force will therefore be, § 64, inversely as the cubes of the radii, or distances from the axis; it may, therefore, in approaching the vein, give the particle a force that will enable it to resist the action of the descending particles of the

vein that it tends to join. The greater the height whence the latter have descended, the greater will be their action to destroy the centrifugal force, which will be constant; and hence, the cavity that will be formed by the latter, on the surface of the liquid, will increase as the depth of liquid in the vessel diminishes. The figure of the section of the cavity has been investigated on these principles by Venturi, and has been shown to be a curve convex towards the axis of the vein. The curve is one of the third order.

CHAPTER III.

Of the Discharge of Liquids through Short Pipes or Adjutages.

403. When a liquid, instead of flowing through an orifice placed in a thin plate, issues by a short pipe; and if the pipe and the liquid be of such materials as will act mutually upon each other by the attraction of cohesion, the edges of the orifice will exert a force upon the filaments of liquid in contact with them; the consequence of this should be their deviation from their original direction towards the vertical, and a consequent increase in the dimensions of the vena contracta.

This theory is fully confirmed by experiment, whence it appears, that the adaptation of pipes, to the orifices whence liquids issue, increases the quantities discharged. Such additional tubes are called Adjutages.

404. Adjutages of different forms, have different degrees of advantage in this respect, that can only be determined by experiment.

When a cylindrical tube is adapted to a circular orifice, the discharge is increased, until the length amount to four times its diameter; after this limit, it again decreases in consequence of friction in the tube.

The increase in the discharge is in the ratio of 82 : 62.

The same increase still takes place, if the tube be contracted in the form of a frustum of a cone, whose altitude is at the distance of half the radius from the orifice, and whose lesser base has an area of $\frac{62}{100}$, of the area of the orifice; and again spread out to its original size, by the adaptation of another conic frustum.

If the first cone be merely inserted in a cylindric tube, the increase is only in the ratio of 77 : 62.

If the tube have the form just described of two truncated cones, adapted to each other at their lesser bases; the greater bases having the same area with the orifice, the lesser that of the vena contracta, or $\frac{62}{100}$; the cone next the orifice an altitude equal to its radius, the other cone an angle at the vertex of 36°, the discharge is increased in the ratio of 92 : 62. If the latter cone be lengthened until the area of its greater base becomes one half more than that of the orifice, the discharge is increased in the ratio of 94 : 62.

Thus it appears that in an adjutage, a part of which is contracted to the area of the vena contracta, or to no more than .62 of the orifice, the actual discharge may approach within .06 of the theoretic.

If then there exists a right to draw water through a pipe of given dimensions, the quantity determined by theory may be increased in the ratio of 132 : 100, by merely uniting the pipe to the receiver by a truncated cone, the area of whose lesser base is that of the pipe; whose larger base has an area that bears to the less, the ratio of 100 : 62; and whose altitude is half the diameter of the greater base. A near approach is obtained to this form, by making the diameters of the base as 10 : 8, and the height of the cone $\frac{5}{8}$ths of the diameter of the tube.

A still farther increase in the ratio of 150 : 100 may be obtained by making the tube spread out at its place of discharge, in a conical form, at an angle of 16° with its axis; and this increase will be obtained, even if a cylindrical tube of considerable length intervene between the two cones.

It has been stated by some writers that this last increase will take place, whatever be the length of the intervening cylindrical tube. But this is not the case beyond that limit at which the velocity of the water in the pipe becomes constant, or when the retarding and accelerating forces counteract each other.

Similar results take place in channels of forms other than cylindric; in them all, an increase of the liquid they will carry, may be effected by giving the tubes the forms the liquid would assume under the mutual action of its particles.

405. These results, in the case of tubes of circular section, are very remarkable, and are worthy of exhibition in a tabular form.

TABLE

Of the quantities of a liquid discharged in equal times from adjutages of different figures.

From an orifice in a thin plate, ...	0.62
Through a short cylindrical tube, whose area is the same as that of the orifice, . . .	0.82
Through a tube contracted at the distance of half its diameter from the orifice to an area of .62,	0.82
Through a tube of the figure of two truncated cones, whose least base is .62, and whose two greater bases are equal to the orifice, .	0.92
Through a tube formed of similar cones, whose length is increased until the area of its place of discharge, is one half greater than that of the orifice, ...	0.94

Theoretic discharge, ...	1.00
Discharge through a given aperture, connected with the reservoir by a truncated cone of which the aperture is the lesser base, . .	1.32
Discharge through the same aperture, connected with the reservoir in the same manner, and which has another truncated tube adapted to it, the angle of whose vertex is 32°, . .	1.50

CHAPTER IV.

Of the Motion of Water in Pipes.

406. When the tube that is adapted to an orifice, by which water flows from a reservoir, is of a length greater than four times the diameter of the orifice, the velocity is retarded; this retardation is caused by a resistance, arising from the friction of the liquid against the sides of the tube. Under the action of this resistance, the velocity of the liquid, which at first varies with the square root of its depth, will finally become constant.

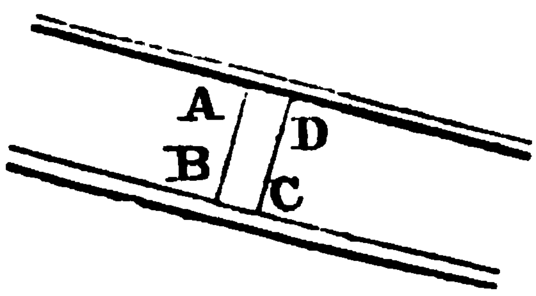
The law which this resistance follows, has been determined by experiment. It has been thus found to be a function of the velocity, and of such a nature that it may be conveniently divided into two parts; the first of which is directly as the velocity; the second directly as its square. The resistance also varies with the surface, by which the liquid is in contact with the channel in which it runs.

To express this law analytically:
Let v be the velocity;
α, and β, constant co-efficients, determined by experiment;
s the surface;
The friction, f, will be

$$f=s\,(\alpha v+\beta v^2) \qquad (417)$$

When a liquid moves in a tube of uniform diameter with a constant velocity, the current fills the whole of the tube; the particles of the liquid in immediate contact with the tube, will be most resisted by friction; but in consequence of the viscidity common to all liquids, they will receive motion from the neighbouring particles, and will in turn retard them; hence, although the velocity of all the particles situated in a given transverse section of the tube is not constant, it may, without any sensible error, be considered as such.

Let us then suppose that the space occupied by a liquid, that has acquired an uniform velocity in a tube, is divided into a great number of layers, infinitely thin, by means of planes perpendicular to the axis of the tube. Let A, B, C, D, be one of the layers into which the fluid is divided. The motion being uniform, the resultant of all the forces that act upon it is $=0$, or, § 39, they are in equilibrio. Among the forces that accelerate are the fluid pressures; if that upon the unit of surface of the face AB,



be p, that on the unit of the face, C, D, will be

$$p+dp.$$

The forces which oppose the motion of the liquid will be:

(1). The difference of the fluid pressures on the opposite surfaces of the layer. If a be the area of these surfaces, this resistance will be

$$a\,dp.$$

(2). The friction. This as has been seen, (417) will be represented by

$$s\,(\alpha v+\beta v^2);$$

but in a thin layer, we may substitute the circumference, c, of the pipe, multiplied by the differential of the length l, for s, and this resistance will become

$$c\,(\alpha v+\beta v^2)\,dl. \tag{418}$$

On the other hand, the force which tends to move the liquid in the tube, is that component of its weight which lies in the direction of the axis of the tube.

Let i be the inclination of this axis to the vertical; the whole weight of the layer will be

$$a\,dlg;$$

and its component in the direction of the axis of the tube,

$$a\,dlg\cos.\,i.$$

If the difference of level of the points, A and C, be dz, we have

$$dz=dl\cos.\,i, \tag{419}$$

therefore

$$a\,dlg\cos.\,i=agdz. \tag{420}$$

When the motion is constant, § 39, this force must be in equilibrio with the two first, or

$$a\,gdz=a\,dp+c\,(\alpha v+\beta v^2)\,dl. \tag{421}$$

Integrating, and introducing for the constant quantity, the initial pressure at the origin of the tube, P, we have

$$a\,gz=a(p-\mathrm{P})+c(\alpha v+\beta v^2); \tag{422}$$

which, when l becomes equal to the length of the tube, becomes, calling the pressure at its place of discharge, P′,

$$agz=a(\mathrm{P}'-\mathrm{P})+c(\alpha v+\beta v^2)l; \tag{423}$$

whence we obtain

$$\alpha v+\beta v^2=\frac{a}{c}\,\frac{gz-(\mathrm{P}'-\mathrm{P})}{l}. \tag{424}$$

If the diameter of the tube be D,

$$\frac{a}{c}=\tfrac{1}{4}\,\mathrm{D},$$

and

$$\alpha v+\beta v^2=\tfrac{1}{4}\,\mathrm{D}\,\frac{gz-(\mathrm{P}'-\mathrm{P})}{l}. \tag{425}$$

If H be the column of liquid that presses on the origin of the tube, and H′, that which presses on its place of discharge, we have

$$P=gH,\ P'=gH';$$

and substituting we obtain

$$\alpha v+\beta v^2=\tfrac{1}{4}\,Dg\,\frac{z-H'+H}{l}. \qquad (426)$$

If the origin of the ordinate, z, be taken at the surface of the column, H,

$$z-H'+H=H-H'; \qquad (427)$$

and if the discharge take place in the open air,

$$z-H'+H=H;$$

in which case

$$\alpha v+\beta v^2=\tfrac{1}{4}\,Dg\,\frac{H}{l}. \qquad (428)$$

By the researches of Prony, who compared fifty different experiments made on tubes, for conveying water,

$$\frac{\alpha}{g}=0.00017,$$
$$\frac{\beta}{g}=0.003416,$$

therefore, the quantities being estimated in metres,

$$0.00017\,v+0.003416\,v^2=\frac{1}{4}Dg.\frac{H}{l}. \qquad (429)$$

Whence we obtain, by taking a value for v, deduced in a particular case from experiment,

$$v=-0.0248829+\sqrt{\left(0.000619159+717.885\,D^2\,\frac{H}{l}\right)} \qquad (430)$$

which becomes, when the quantities are estimated in English feet,

$$v=-0.1541+\sqrt{\left(0.02375+8201.6\,\frac{DH}{l}\right)}. \qquad (431)$$

The co-efficients α and β, become, if applied to the English foot as the unit,

$$\alpha=0.00017,$$
$$\beta=0.000104;$$

and neglecting the term that involves the velocity simply; which,

as the co-efficient is small, may be done, if the velocity be not great, without any sensible error, we have

$$0.000104\, v^2 = \frac{1}{4} D \frac{H}{l}, \tag{432}$$

and

$$v^2 = 2404\, D \frac{H}{l}; \tag{433}$$

$$v = 46.82 \sqrt{\left(D \frac{H}{l}\right)}. \tag{434}$$

If the diameter of the tube be estimated in English inches, as is most usually the case, and the other quantities in feet,

$$v^2 = 200\, D \frac{H}{l}; \tag{435}$$

and

$$v = 14.142 \sqrt{\left(D \frac{H}{l}\right)}. \tag{436}$$

If Q represent the quantity discharged by the tube in the unit of time

$$Q = v \frac{\pi D^2}{4}, \tag{437}$$

and

$$v = \frac{4Q}{\pi D^2}; \tag{438}$$

substituting this in the equation (428), and neglecting the first power of v, we have

$$b \frac{16\, Q^2}{\pi^2 D^4} = \frac{1}{4} D \frac{H}{l}; \tag{439}$$

and if we make

$$b = \frac{64\, \beta}{\pi^2},$$

we have

$$bQ^2 = D^5 \frac{H}{l},$$

and

$$Q = \sqrt{\left(\frac{D^5 H}{b\, l};\right)} \tag{440}$$

calculating the numeric value, we obtain

$$b = 0.0000688,$$

and

$$Q = 38.13 \sqrt{\left(D^5 \frac{H}{l}\right)}, \tag{441}$$

in which all the lineal dimensions are in English feet, and the quantity in cubic feet.

For the quantity in cubic feet, when the diameter D of the tube is given in inches, we have

$$Q=0.0778\sqrt{\left(D^5\frac{H}{l}\right)} \qquad (442)$$

For the quantity in English statute gallons, D being in inches as before,

$$Q=.48635\sqrt{\left(D^5\frac{H}{l}\right.}. \qquad (443)$$

For the quantity in standard liquid gallons of the State of New-York.

$$Q=.608\sqrt{\left(D^5\frac{H}{l}\right)}. \qquad (444)$$

The formulæ for the value of D, when Q, H, and l, are given, can be easily obtained from the foregoing, the fundamental expression being,

$$D=\sqrt[5]{\left(Q^2\frac{l}{b}\right)}. \qquad (445)$$

The formula (434), is similar to that of Prony, for metres, which is

$$v=26.79\sqrt{\left(D\frac{H}{l}\right)}, \qquad (446)$$

but which reduced to English measure, would be

$$v=48.5\sqrt{\left(D\frac{H}{l}\right)}, \qquad (447)$$

the co-efficient being 48.5, instead of 46.82, as we have made it.

Prony's formula, however, appears to be in excess, except when the velocity is considerable; that of (434) is probably in defect, except at small velocities.

The formula of Du Buat, who led the way in these investigations, is

$$V=\frac{307\sqrt{(d-0.1)}}{\sqrt{s}-\log.\sqrt{(s+1.6)}}-0.3\sqrt{(d-0.1)} \qquad (448)$$

In which,

V is the velocity in English inches;

d half the radius of the pipe;

s the mean slope of the pipe which is equivalent to $\frac{H}{l}$ of Prony's formula.

log. The hyperbolic logarithm of the quantity to which it is prefixed.

The computation by this formula may be facilitated by means of subsidiary tables, the best set of which are to be found in the Edinburgh Cyclopedia, article, Hydrodynamics.

The formula of Eytelwein, is in English feet.

$$v=50\sqrt{\left(\frac{DH}{l+50D}\right)}. \qquad (449)$$

407. The foregoing investigation is only applicable to the case of a pipe of uniform slope, lying in the same vertical plane, and of a constant section. If the diameter be not constant, the discharge will obviously be due to the area of its least section; but will be affected by the same causes that influence the discharge of fluids through orifices and adjutages. The velocity may be calculated as above, and be multiplied by the smallest area of the pipe. The amount thus obtained, must then be increased or diminished by the use of the co-efficient, which may be obtained from the table in § 405, according to the nature and form of the contraction:

The quantity D, used in the calculation of the velocity, must be the general diameter of the pipe, for the friction will obviously be principally due to it, or nearly so, and not to those portions that are contracted.

The necessity of continuing a pipe of uniform bore, from the place where it receives the liquid it is to carry, to the place where it is to discharge, is therefore manifest; but at its two extremities it should have conical adjutages.

408. Pipes that convey water, are liable to two species of obstruction, that tend to diminish the effective areas of their section, and thus lessen the quantity they would otherwise discharge. All spring or river water contains gaseous matter: this will often escape and separate itself in consequence of its expansive force; hence lodgments of air may take place in the higher parts of the pipe, and where the pipe, after having risen, is bent, and again descends. A self-acting apparatus has been planned to permit the escape of such lodgments of air. It consists of a valve of the form of a sphere, that is placed in a vertical cylinder, adapted to the upper bends of the pipe; in the cap that closes this cylinder, a hole is cut for the seat of the valve, and is carefully ground to the figure of a hollow zone, of a sphere of the same radius as the spherical valve. The valve is made of metal, and is hollow, in order that it may be light enough to be buoyant in water. When the pipe runs full of water, the pressure of the liquid keeps the sphere closely applied to its seat; but when a lodgment of air takes place, the sphere falls; the valve is therefore opened, and the air escapes; the water which follows lifts the sphere, and applies it to the seat that has just been described. The only precaution in using this is, to take care that the valve seat shall be lower than the level of the water in the reservoir whence the pipe is supplied.

A simple stopcock, that is occasionally opened, will answer the same purpose, but is not self-acting.

The pipe may be interrupted at the places where the air is likely to lodge and the water discharged into a basin, whence it is again drawn by the prolongation of the pipe. In this case, all the advantage derived from the superior height of the water in the original reservoir is lost. This may be obviated by raising the pipe in this place, by artificial means, to the height due to the velocity of the liquid, which will be as much less than the original head, as is due to the friction.

Such an arrangement is called a Souterazi.

It possesses, when applied to very long lines of pipes, an important advantage; for the water conveyed in them becomes vapid and disagreeable, but will, by exposure to air, recover its qualities.

Deposits of earthy matter often take place in the lower angles of a pipe. These arise from substances that are either mechanically mixed, or held in solution in the water. These deposits may be removed by throwing a cork, to which a string is attached, into the pipe. If the space left in the pipe be sufficient to admit its passage, it will carry one end of the string to the place of discharge, and an instrument for cleansing the pipe adapted to the other end, may be drawn through the pipe by means of it.

Stopcocks may be adapted to the lower angles of the pipe, and opened at proper intervals; the current they cause will carry with it any earthy matter that has not become indurated.

Short tubes may be placed beneath the lower angles, communicating with the pipe, by means of a smaller vertical pipe. The deposit will take place in them, instead of the main pipe, and they may be removed as often as necessary, and cleansed.

409. If the pipe be not of uniform slope, or do not lie wholly in the same vertical plane, the water moving in it will experience a resistance at the angles. The amount of the resistance has been ascertained by the experiments of Du Buat. He found it to be proportioned to the square of the velocity, and to the square of the sine of the deviation of the tube from its original direction, to the number of bends or elbows in the tube.

For a single angle, therefore, this resistance may be thus expressed:

$$R = mv^2 \sin.^2 i,$$

and for any number of elbows,

$$R = mv^2 \Sigma . \sin.^2 i. \tag{450}$$

The co-efficient, m, as determined by Du Buat, is in French inches,

$$m = \frac{1}{2998.5} = 0.000336,$$

in metres,

$$m = 0.0123\ ;$$

and in English feet,

$$m = 0.039.$$

In the latter case the formula (450) becomes

$$R = 0.0039 v^2 \Sigma . (\sin.^2 i)\ . \qquad (451)$$

This formula ceases to be true when the angles of deviation exceed 30°.

410. The general formulæ, (428) and (440), for the velocity and quantity discharged by a pipe, are only applicable to the case of a single pipe of uniform bore throughout, or where there are a few definite contractions in the course of such a pipe. They are not adapted to the circumstances of lateral tubes that diverge from a main pipe, for the purpose of distributing a liquid to different points, as is frequently necessary in the supply of cities with water. Into these the water will enter with a velocity, that is due to its pressure upon the part of the main pipe, to which the lateral tube is adapted.

Water, as may be inferred from the preceding investigations, moves in a tube in consequence of the pressure upon its origin, and the weight of the particles in the descending branches, and is resisted by the friction against the pipe, and the weight of the particles in the ascending branches. One part of the moving power is, therefore, employed in generating the velocity of the liquid; another in overcoming friction; while the third is expended upon the resistance of the columns that act in a direction contrary to the motion. The last is the principal element that determines the pressure on the tube. The main tube may therefore be considered as a reservoir whence the lateral pipe is withdrawn, and all the circumstances determined upon the principles that we have applied to the main tube, and its reservoir. The pressure on any given point will, theoretically speaking, be due to the difference between the actual height, and that due to the velocity of the liquid. This pressure cannot be always practically determined with an accuracy sufficient for the purpose. It will be seen, however, in the following investigation, that the discharge of the lateral pipes may be determined by means of expressions, into which the pressure does not enter.

Let Q, be the quantity of water the main pipe would deliver at the point whence the first lateral pipe diverges;

D, the diameter of this pipe ;

L, its whole length ;

$\lambda\ \lambda'\ \lambda'' \ldots \lambda^{n-1}$, the partial lengths of the main pipe, whose sum is equal to L ;

Z, the difference of level between the water in the reservoir, and the opening of the first lateral tube ;

$Z'\ Z'' \ldots\ldots Z^{n-1}$, the difference of level between each two consecutive branches.

$H'\ H''\ H''' \ldots\ldots H^n$, the heights due to the pressures on the points whence the successive branches diverge ;

$$\left.\begin{matrix} q' & d' & l' & z' \\ q'' & d'' & l'' & z'' \\ q^n & d^n & l^n & z^n \end{matrix}\right\}$$ similar elements for each separate branch ;

c, the constant quantity in the formulæ (440), to (444), according to the measure employed; we have for the several parts of the main pipe, and its branches from (440),

$$\left.\begin{array}{lr} Q=c\sqrt{\left(\frac{Z-H}{\lambda}D^5\right)} & (a) \\ q'=c\sqrt{\left(\frac{H'-z'}{l'}d'^5\right)} & (b) \\ Q-q'=c\sqrt{\left(\frac{H'+Z'-H''}{\lambda'}D^5\right)} & (c) \\ q''=c\sqrt{\left(\frac{H''-z''}{l''}d''^5\right)} & (d) \\ Q-q'-q''=c\sqrt{\left(\frac{H''-Z''-H'''}{\lambda''}D^5\right)} & (e) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ q^n=c\sqrt{\left(\frac{H^n-z^n}{l^n}d^{n5}\right)} & (f) \end{array}\right\} \quad (452)$$

The equations being in all to the number of $2n$, or twice as many as there are lateral pipes.

To eliminate H′, H″, &c., we first combine the equations (a) and (c), (a), (c), and (e), &c. ; we thus obtain

$$Q^2\lambda+(Q-q')^2\lambda'=c^2D^5(Z+Z'-H''), \qquad (g)$$

$$Q^2\lambda+(Q-q')^2\lambda'\ldots\ldots+(Q-q'\ldots\ldots q^{n-1})^2\lambda^{n-1}= c^2D^5(Z+Z'\ldots\ldots Z^{n-1}-H^n). \qquad (h)$$

Then by combining the equations (a) and (b), (d) and (g), &c. we have

$$Q^2\lambda d'^5+q'^2l'D^5=c^2D^5d'^5(Z-z'), \qquad (i)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$[Q^2\lambda+(Q-q')^2\lambda'\ldots\ldots+(Q-q'\ldots-q^{n-1})]d^{n5}+q^n2l^nD^5= c^2D^5d^{n5}(Z+Z'\ldots\ldots+Z^{n-1}-z^n). \qquad (k)$$

From the last equation we may obtain the values d' d'', to d^{n-1}, which are

$$\left.\begin{aligned} d' &= \sqrt[5]{\left(\frac{q'^2 l\, D^5}{c^2 D^5 (Z-z) - Q^2\lambda}\right)} \\ d'' &= \sqrt[5]{\left[\frac{q''^2\, l''\, D^5}{c D^5 (Z+Z''-z'') - (Q^2 + (Q-q')^2\lambda)}\right]} \end{aligned}\right\} \quad (453)$$

These two equations are sufficient to determine the law; and in order that they shall give rational values for the diameters of the lateral pipes, it is necessary that the denominators of the fractions should be positive*.

*See, "*Essai sur les Moyens de conduire d'élever et de distribuer les Eaux*, par M. Genieys.

CHAPTER V.

Of the Motion of Liquids in open Channels.

411. When a liquid issuing from a reservoir enters into an open channel, the general direction of the bed must be inclined downwards, otherwise it would not continue to flow; and the surface will have a slope from the reservoir towards the place of discharge. Being acted upon by the force of gravity, the liquid will have a tendency to assume an accelerated velocity. It rarely however happens, and only when the slope is very great, or the length of the channel small, that this acceleration does occur; in some cases the velocity, so far from increasing with the distance from the reservoir, or source, diminishes. In most instances the mean velocity is found for long distances, to be uniform, and to change only with changes in the nature and character of the bed.

When a stream flows in a channel with uniform mean velocity, it is said to be *in train*; this can only occur when no accelerating force acts, or when the sum of the accelerating and retarding causes is $=0$.

The circumstances, then, of the uniform motion of a liquid in an open channel of uniform section, may be made the basis of the theory of the motion of liquids in open channels of any figure or variety of dimension whatsoever; and the variations from the simple theory which the change of dimension may produce, can, if necessary, be applied as corrections to the inferences.

The principal retarding force, to which the motion of liquids in open channels is liable, is the friction upon their beds. This, according to the experiments of Coulomb, will be a function of the velocity and of the surface directly, and be inversely as the area of the section of the stream; and as in the case of tubes, that part of the resistance that is à function of the velocity, has two terms; in one of which the first, and in the other, the second power of the velocity are involved. This observation forms the basis of the theory.

This retarding force may be thus expressed:

$$f=g\,\frac{s}{\omega}\,(\alpha v+\beta v^2)\,. \qquad (454a)$$

In which formula α and β are constant co-efficients, determined in some particular case by experiment;

f, the friction;

g, the measure of the gravitating force;

s, the length of the perimeter of that surface, which is in contact with the liquid;

ω, the area of a transverse section.

Now let

$l=$ the length of the axis of the channel;

$\mathrm{H}=$ the difference of level between the places where the channel receives and delivers the liquid; we may obtain by a course of reasoning similar to that in § 402.

$$\alpha v+\beta v^2=\frac{g\omega \mathrm{H}}{ls}=g.\frac{\omega}{s}.\frac{\mathrm{H}}{l}.$$

The quantity $\frac{\omega}{s}$, is that which is called the Hydraulic mean depth, or Radius; this is usually represented by R. The quantity $\frac{\mathrm{H}}{l}$, is the slope or inclination of the tube, which is represented by I. Using these symbols, we have

$$\alpha v+\beta v^2=g\mathrm{RI}. \tag{454}$$

According to the investigations of Prony, the constant numbers applicable to this equation, are, after division by g, when the measures are taken in metres,

$$\alpha=0.0004445,$$
$$\beta=0.0003093;$$

or in English feet,

$$\alpha=0.0004445,$$
$$\beta=0.0000912;$$

and neglecting αv, we have

$$.0000912v^2=\mathrm{RI}; \tag{455}$$

whence

$$v^2=12000\ \mathrm{RI},$$
$$v=109.53\sqrt{\mathrm{RI}}.$$

Eytelwein makes the numbers for English feet,

$$\alpha=0.000243,$$
$$\beta=0.00001113;$$

whence we obtain, again neglecting αv,

$$v=94.87\sqrt{\mathrm{RI}}. \tag{456}$$

If we employ the same constant number for β, that we have made use of in tubes, § 402, we have

$$v=93.64\sqrt{\mathrm{RI}}; \tag{457}$$

or taking the simpler formula of Prony, as in (447),

$$v=97\sqrt{\mathrm{RI}}. \tag{458}$$

By the application of experiment to determine the value of v, in a particnlar case, Prony obtains a general formula, applicable

both to tubes and open channels; this is as follows, the measures being in English feet:

$$v = -0.154 + \sqrt{(0.0238 + 32806\,G)}. \qquad (459)$$

In this formula, in the case of pipes,

$$G = \tfrac{1}{4} D \frac{H}{l}; \qquad (460)$$

in the case of open channels,

$$G = RI = \frac{\omega}{s} \cdot \frac{H}{l}. \qquad (461)$$

In the former case, therefore, it is identical with (431).

412. The velocity, v, in the foregoing investigations is the mean velocity, and will not be that of all parts of the stream. Those portions which are nearest to the sides and bottom of the channel, will be directly resisted by them; and although in consequence of the viscidity, they will derive motion from those more distant, and will retard them, the latter will have the greater velocity. On the other hand, the liquid pressure will tend to accelerate the threads of fluids which lie deepest beneath the surface of the stream. By the combination of these two actions, the greatest velocity in a symmetric open channel will be in its middle, and at a small distance below the surface. Between the greatest velocity at or near the surface, and the mean velocity, there is a constant relation, as has been shown by the experimental researches of Du Buat. The formula derived by him from his experiments is however faulty, and has been converted by Prony, into the following form:

Let v, be the mean velocity;

V, the greatest velocity;

$$v = \frac{V(V+a)}{V+b}. \qquad (462)$$

From this, may be derived the following mean value of v,

$$v = 0.816458\,V; \qquad (463)$$

or, what is in most cases sufficiently accurate,

$$v = \frac{4}{5} V. \qquad (464)$$

413. When the channels in which streams run, are neither of uniform section throughout, nor directed in a straight line, variations from the above theoretic inferences take place.

Bends and elbows in the stream obstruct its course, and diminish the discharge; the current will in such cases be directed from one of the points towards the next, which it will tend to wear away, and in the bays that intervene, eddies take place.

The formation of eddies may be thus explained. In conse-

quence of the viscidity of liquids, a current that is in motion, tends to carry with it the neighbouring masses of a similar fluid nature; thus, if a current be passing through a space greater than it is capable of occupying, without changing its velocity, the lateral fluid will join the stream, and tend to increase its quantity; by this flow the level will be lowered, and the pressure of the adjacent masses will impel a current towards the lowest point, which will be that whence the fluid is first drawn. In a wide channel, the main stream will be increased by this action, and on one or both sides of it, a current will be formed in an opposite direction, into which, when the space is again contracted, or another point reached, the excess of fluid that has united itself with the main stream will flow, and thus keep up the circulation. If, on the other hand, the increase in the section of the channel be permanent, eddies will at first be formed, but the velocity of the main stream will gradually diminish, until it become capable of filling its new bed.

CHAPTER VI.

Of Rivers.

414. The vapour which is raised from the whole surface of the earth, and particularly from the ocean, tends to distribute itself uniformly according to the mechanical law, mentioned in § 355, and known by the name of its discoverer, Dalton. Thus, the excess of moisture furnished to the atmosphere, from the surface of the sea, is borne towards the land; upon the latter, as will hereafter be shown, the causes that produce precipitation are, in general, more frequent than on the ocean; hence, more moisture falls on the surface of the land, than is evaporated from it. The loss arising from this excess of precipitation is again supplied from the ocean, in conformity with the law we have cited. It thus happens that in most parts of the continents and islands, the quantity of moisture that falls to the surface, in the form of rain, hail, snow and dew, exceeds that which is evaporated from their surface. This excess partly runs over the surface of the ground, and partly penetrates into it. In the latter case it frequently meets impervious strata; along these the water is carried by its gravity, until they break out to the day. The water will there form springs. These unite with that running upon the surface, and descend to the lowest points of vallies, where they form lakes, or, if the slope be sufficient, streams of various magnitudes. Lakes when they increase to such a height as to overtop the lower parts of their barriers, or acquire a sufficient pressure to force their way through them, also give rise to streams. Such streams running in the lower parts of vallies, and upon the lines of the greatest slope, unite at the junction of two or more vallies, and mix their waters in a channel of greater magnitude; and thus, by successive junctions, if the extent of country be sufficient, rivers of great magnitude are formed.

415. The formation of the beds and channels of rivers appears, even on the most cursory observation, to have been effected by the action of their own waters: from the continued effort of their streams, a species of equilibrium has taken place between the motion of their currents, and the resistance of the soil over which they run. Thus: when the velocity is very great, the beds are composed of solid rock; in the next stage of diminishing velocity, the beds are composed of large rolled stones; when the velocity decreases still farther, the bottom is composed of gravel; then of

sharp sand; and finally, when the water becomes stagnant, of fine argillaceous particles, or mud.

Still the equilibrium between the active force of the stream and the resistance of its bed, is not absolutely perfect; a slow and gradual action takes place on all parts of its bed, imperceptible, except in the larger class of streams, or under extraordinary circumstances; but this action is sure, and after the lapse of years will be found to have produced most important effects.

It has been determined by the observations of Du Buat, that fine clay will not resist the action of a current, whose velocity, at bottom, exceeds three inches per second: fine sand begins to resist when the velocity falls below six inches; coarse sand, at a velocity of eight inches; gravel, at velocities from seven to twelve inches; pebbles of an inch in diameter, at two feet; and angular fragments, of the size of an egg, at three feet per second.

Rivers cannot be considered as forming, throughout their whole course, channels of uniform slope; on the contrary, they are, as a general rule, most inclined near their sources, and become less and less so as they approach the sea. They may, however, in most cases, be divided into portions; each of these may be considered as composed of a current flowing with a constant area, and having an uniform slope at its surface. These portions are frequently separated by marked physical barriers.

Under the long conflict of the active forces of running water, and the passive resistance of their beds, the latter have, in almost all cases, assumed a state of apparent permanence; the changes that take place in them are slow, and only perceptible by the comparison of their states at long distant periods.

416. Rivers are subject to an increase and diminution in the quantity of their waters. This variation sometimes bears but a small proportion to their mean magnitude, at others is of great extent. It is in some cases periodic, varying with the season, being produced by the melting of the snows, in other cases, is subject to no fixed law.

When the variation in the bulk of a stream is small, its bed has usually for its section a concave curve, whose versed sine bears a greater ratio to its chord, when formed in earth that opposes a great resistance, and of course, when the stream is rapid, than it does in soils of less tenacity. The resistance of solid rock may render this rule untrue when the bed is formed in that substance.

When the variation in the bulk of the stream is great, and its slope small, there is usually a bed of the same form as in the other case, suited to convey the stream when it does not much exceed its mean magnitude; this is enclosed on one or both sides by the

alluvial deposits of the river. These usually assume the shape of an inclined plane, sloping from the bank of the main stream towards the adjacent country. The manner in which this shape is produced, may still be witnessed in the streams of many parts of our own country. The river, as its volume increases, flows with greater rapidity, and carries with it an increased quantity of earthy matter; when it rises so far as to overtop the banks which bound it when not swollen, if trees and bushes grow upon them, they will catch and retain the larger and heavier parts, while the lighter alone will remain suspended. Thus the greater proportion of the deposit takes place on the edge of the ordinary bed. This process may still be annually witnessed on the Mississippi; and the height of the natural dyke that is thus formed, is increased by the quantity of drift wood that is retained by the nearest obstacles. This natural barrier, so long as the periodic overflow is unimpeded, more than counteracts any rise that may take place in the bed, from deposits at those times when the stream has less than its mean velocity.

It frequently however happens in cultivated countries, that it becomes necessary to restrain the periodic overflow. For this purpose dykes are erected on the borders of the usual channel, their erection being facilitated by the natural form of the adjacent ground. In this case, the substances carried by the stream are deposited in its bed, instead of being spread over the whole surface, to which the inundation had before extended. The level of the surface of the stream will in consequence rise, until the slope may finally become too small to convey it, and it may seek an outlet in other directions. It is thus rendered indispensable to raise the dykes to correspond to the increased height of the surface of the stream. By a long continued process of this kind, the bottom of the bed of the Po has been raised higher than the level of the adjacent country, and the surface of its stream overtops the houses of Ravenna; so also the ancient channel of the Rhine has been filled up, until it will no longer carry its waters, and they have sought outlets in other directions.

A similar action is going on upon the main outlet of the Mississippi, where the superior magnitude of the stream makes a change in its bed, and in the height of its waters, when full, more marked in a few years than it is in the Po and Rhine in centuries.

A river that requires to be confined by dykes, is often crooked in its course. The danger of overflow may, in this case, be lessened upon principles derived from our investigations. It will be seen by reference to the formula (456), that the velocity varies with the square root of the slope; if then the course be rendered

straight, the fall between two given points remaining constant, while the distance in the direction of the stream is lessened, the slope is augmented, and a bed of given dimensions will carry a greater quantity of water, in a given time.

417. Among the many important purposes that rivers subserve, that of navigation is worthy of particular notice.

Rivers, according to their size and importance, are sometimes navigable for vessels of various descriptions and sizes; but are, at others, frequently liable to obstructions and impediments, that either interrupt or prevent their navigation altogether.

These obstructions and impediments may be arranged in several distinct classes.

(1.) Rivers may be obstructed by Falls or Cataracts, consisting in a sudden change of level.

(2.) Rocky barriers may cross the bed of a river; these will prevent its forming a channel sufficiently large to discharge its waters with its usual mean velocity. In this case, the flow of the higher part being impeded, the water accumulates above the barrier, until the slope becomes so great immediately over it, as to increase the velocity there to an extent adapted to the discharge of the stream, through the contracted space. Such obstructions are called Rapids.

In large streams the rocky barrier sometimes lies so deep that navigation is not obstructed, but merely embarrassed, by a current of increased velocity. Such is the rapid at the outlet of Lake Erie, and a more magnificent instance is to be seen in the *Race* in Long Island Sound.

In small streams they interrupt the navigation altogether.

(3.) When rivers, after running in a mountainous country, reach one of less slope, their velocity is diminished; the earthy particles they had before been able to carry with them, are in consequence deposited, the bed is filled up, and the river seeks a discharge by spreading itself. Thus the breadth of the channel is increased, and its depth diminished. A similar consequence may follow when the velocity of a river is lessened by its meeting the tide within its own channel. By a combination of these two causes, the obstructions that exist in the Hudson River, near Albany, have been formed.

(4.) Where a river that carries large quantities of earthy matter enters the ocean, the conflicting action of the two masses of water, causes a cessation of motion that influences a deposit; bars are thus created in the sea, in advance of the mouth of the river. In the case of large rivers, these may become the basis of islands, which are gradually connected with the main land, while the force of the current sweeps out the intervening channels, and new bars

are formed beyond them. In this manner the Deltas of rivers were originally deposited, and, in some instances, still continue to protrude themselves into the sea.

(5.) A river may carry a sufficient quantity of water to admit of navigation, and may be unobstructed by falls, rapids, or shallows, but may be so rapid as either to prevent an ascending trade, or to diminish the area of the stream so far as not to admit of a vessel floating in it.

418. Each of these obstructions has its appropriate remedy, the application of which may however be, in some cases, impracticable, either from physical circumstances, or the great expenditure they involve.

(1.) When a river is obstructed by falls, we make a lateral channel, fed from the upper level, and apply locks to it, upon principles that will be explained under the head of canals.

(2.) If rapids have upon them a sufficient depth of water to float the vessels, artificial mechanical means may be brought in aid—thus: the power of men or of animals may be applied from the shore; powerful steamboats may be used in towing; or the vessel may have wheels adapted to it, the area of whose paddles is greater than its own section. In this last case, if a barrel be adapted to the wheels, and a rope passed around it and fastened to the shore above the rapid, the wheels being more powerfully acted upon by the current than the vessel is, will turn around, coil up the rope, and drag the vessel forward. It however generally happens that the impediment of a rapid must be overcome by a lateral channel, as described in the preceding instance.

(3.) As the formation of shallows arises from a diminution in the velocity of a stream, it may be prevented, and the obstacle may even be removed, by increasing the velocity. This may be done by contracting the horizontal dimensions of the bed. The upper portions of the stream being thus retarded, the level rises, until the slope, as in the natural formation of a rapid, becomes sufficient for the discharge. With this increased velocity, the stream will have sufficient force to carry away the earthy matter it before deposited, and a permanent improvement in its depth will take place.

As a stream tends to continue of uniform section, even in an increased channel, forming eddies in the bays and hollows, it is not necessary that this contraction should be effected by continuous and parallel lateral dykes. It is sufficient that piers be built out, alternately from each bank, at distances from each other equal to about the breadth it is proposed to give the stream. The heads of these piers should be arranged, if possible, in two parallel straight lines, in order that the stream may assume a straight

course, in which, as has been already explained, it will have the greatest velocity. The deposit, which was before uniformly distributed over the bed, will now take place between the piers. In all such cases, a straightening of the channel is advantageous.

If islands be formed in the shallow parts, as often occurs, all the branches of the river that surround them, except one, should be closed, by weirs that rise to the ordinary level of the stream, and which will therefore permit a discharge through these lateral channels when it is swollen. If there be several islands on either side of the main channel, it may be often advantageous to unite them by a longitudinal dyke, in order to prevent the stream from spreading between them.

(4.) A similar principle will direct the operations intended for the removal of bars at the mouths of rivers. It becomes necessary to give the stream such a velocity as will make its force preponderate over the resistance of the ocean. In large rivers, the extent of the bars puts them beyond the reach of improvement. In small streams, piers may be built out from the main land, from each side of the mouth of the river, gradually inclining to each other. The waters being all confined between them, and acquiring from the contraction an increased velocity, will not only cease to make farther deposits upon the bar, but will carry away so much of it as lies between the piers. The deposit will now take place behind the piers on either hand. Where the direction of the river makes a small angle with the line of the coast, a single pier or jetty may be sufficient. Of this method we have fine instances in the ancient port of Dunkirk, and in the present port of Havre: and similar principles have been successfully applied at Buffalo, on Lake Erie.

(5.) A rapid stream may be rendered navigable by building dams or weirs across it from place to place: between each two of these, the depth of the water will be increased and its velocity diminished; by both of these changes navigation may be rendered more easy. The passage of these dams by vessels, was originally effected in one of two different methods; these require illustration, in consequence of their being the germs of more perfect means, that are still in use. The first of these methods was the Sluice; the second, the Inclined Plane.

To form a sluice, an aperture is left in the mass of each of the dams that divide the river into successive ponds, of different levels. This is for the greater part of the time closed by a gate; and as this gate will have an unequal pressure on its opposite sides, in consequence of the difference of level, it can only be opened by a vertical motion.

In the canals of China, the gate is formed of heavy beams, dropped into vertical grooves, made in the masonry of the dam.

These being piled upon each other, the lower ones are thus loaded with a weight that sinks them to the bottom of the passage, and they oppose an effectual barrier to the flow of the water through them; for the small quantity that may penetrate between their joints, will be wholly unimportant.

When the gate is opened, the water will discharge iteslf with great velocity through the passage. The vessels of the descending trade are committed to this current; and as they would run the risk of being swept into the eddies it will form, men are stationed on each bank, furnished with poles, to keep them in the direction of the effluent stream. By this discharge, the level of the water in the upper pond, is lowered; the velocity of the current that flows through the passage is diminished in consequence. Natural agents, such as the force of men or animals, are then applied to draw up the vessels that carry the ascending trade.

If the change of level be considerable, the danger in passing down, and the force required to draw the vessels up, are both too great. This method, therefore, becomes impracticable. In the infancy of inland navigation, the inclined plane of a rude and imperfect form, was introduced in cases of this sort. In a part of the weir was introduced a mound of masonry, of the form of a triangular prism. Of the two inclined planes that composed its upper surface, one descended to the bottom of the lower, and one to that of the upper level of the navigation, as in the figure beneath.

The vessel, in passing from one level to the other, was first drawn up on one of the inclined planes, and passing the ridge, was resisted in its descent on the other. For this purpose ropes were passed around the vessel, and applied to capstans situated on each side.

The sluice, in its original form, is still used in China, as is the inclined plane. The sluice is also still used in the Low Countries, and in the valley of the Po, in cases where the change of level is not more than two or three feet. In almost all other instances, locks have been substituted for sluices, and several locks, in the cases to which the inclined plane was formerly applied.

These methods of improving the navigation of rivers, have the advantage of requiring a comparatively small original cost;

they may, therefore, be used in countries of thin population, and small wealth. They have the disadvantage on the other hand, that they are liable to injury from floods, while the navigation is itself subject to vicissitudes from variation in the supply of the stream. The passage of vessels through sluices, and over the original form of the inclined plane, is difficult, and requires the application of force of an expensive character. For all these reasons, it has been the uniform result of experience, that in spite of the great excess of the first cost, it is, wherever the capital can be obtained, better to make a canal parallel to the river, than to attempt to improve the navigation in its own bed. The principle on which the construction of canals rests, are contained in the following chapter.

419. When a river is to have its navigation improved, or when water is to be drawn from it for supplying pipes, for irrigation, or any other useful purpose, it is, generally speaking, necessary to ascertain the quantity of water that it furnishes. For some physical investigations, the whole quantity that flows in its bed, may be the object of research. But for mechanical purposes, it is, generally speaking, best to limit the inquiry to the minimum supply, for on this will depend the certainty with which the stream can be depended upon for most practical uses.

In a stream of so small a size that barriers can, without difficulty, be erected across it, a dam that interrupts its course, is constructed at some convenient point. In this a gate is placed, formed of a rectangular frame, and a shuttle that can be raised or lowered at pleasure. The shuttle being closed, the passage of the water is interrupted, and it would rise until it found a discharge over the dam: before it reaches this height, the shuttle is drawn up, and the water passes out. When this discharge is just sufficient to prevent the level of the water from rising farther, but not sufficient to cause it to fall, it will be obvious that the gate discharges the exact quantity of water that the stream furnishes.

This quantity may be calculated on the following principles.—The area of the open part of the gate may be considered as an aperture in a thin plate, and must, therefore, receive a correction for the vena contracta. The velocity, which will vary in every horizontal element of the orifice, will have for its mean, that which is due to the depth of the centre of pressure of the opening beneath the surface of the fluid. This velocity being found by the principles of § 401, and the formula (61),

$$v=\sqrt{2gh},$$

is to be multiplied by the area of the aperture and the constant quantity 0.62, § 408.

In streams whose section has an area of more than two square feet, this method would become expensive and troublesome. To gauge streams of an area of from two to ten or twelve square feet, we have recourse to another method.

The area of the stream is carefully measured. Its velocity may next be determined by the aid of an apparatus, called from its inventor, the Tube of Pitot. A tube ABC is taken and bent until one of its branches is at right angles to the direction of the other. This tube is open at both ends, and is contracted at the opening C of its shorter branch. It is placed in the stream, the branch AB in a vertical position, and the horizontal branch is turned around until the stream enters freely and directly into the orifice E. The water entering the tube with a determinate velocity, has such a force as would cause it to rise above the level of the surface of the stream, to the height whence it must have fallen to acquire that velocity. In the tube, it will be resisted by friction, and the height will be lessened. The height to which it rises may be determined by applying a graduated rod to the outside of the tube, if of glass. If, however, the tube be not transparent, and it is usually metallic, a rod, *bc*, is placed in the vertical branch of such specific gravity as to float on water. This rod being graduated, will mark the rise of the liquid. The corresponding velocity may then be obtained by the usual formula (61)

$$v=\sqrt{2gh},$$

which is sufficiently accurate, except when the velocity is small.

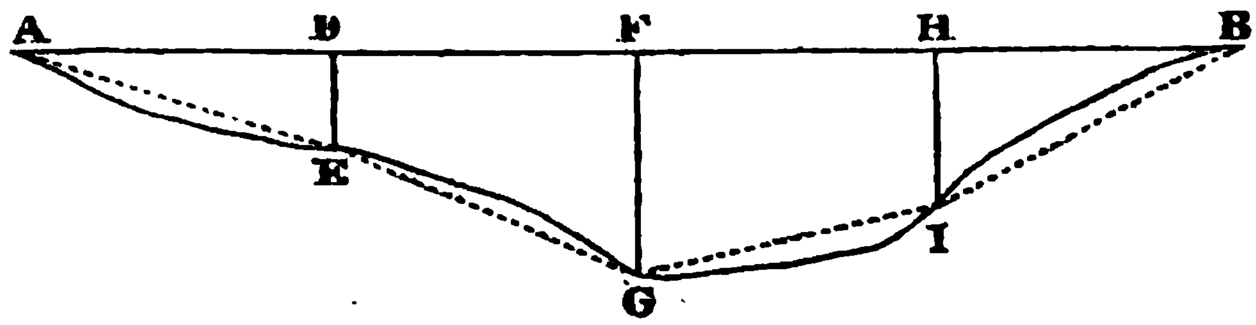
Experiments may be made in different parts of the section of the stream, and the mean of the results taken as the mean velocity. It is, however, generally speaking, sufficient to measure the velocity in this manner, near the middle of the surface, and thence to deduce the mean velocity by the formula (462) or (463).

This velocity, multiplied by the area, will give the discharge in the unit of time.

To use the tube of Pitot, it is convenient to attach it to a wooden stand, by which it can be placed in the bed of the stream and afterwards levelled.

This instrument is, therefore, limited to streams of not more than four or five feet in depth.

In streams of greater size, the velocity of the surface may be measured by a float. For this purpose, a part of the stream must be chosen, where it flows between its banks without eddies. The area of its section may then be determined by measuring its breadth, and its depth, at equal intervals, from bank to bank, as in the following figure, where AB represents the breadth, and



BE, FG, HI, depths measured at three points, dividing the breadth into four equal parts. The area may then be considered as made up of the two triangles, ADE, BHI, and the two trapezoids, DFGE, and FHIG, or of any other number of the latter that the circumstances may require. A float is next thrown into the stream, and the time in which it is carried through a given distance, that is accurately measured upon its banks, noted by a time-keeper. The velocity in the unit of time thence deduced, is reduced, as before, to the mean velocity, and multiplied by the area. The product is of course the discharge. The floats used in this operation, may be constructed in various modes. Thus:

A piece of cork may have a weight attached to it by a string that shall render their united mass but little less dense than water. No more of the cork will then float than is sufficient to render it visible, and it will not be liable to disturbance from the action of the wind. In a very short time it will acquire the whole velocity of the part of the current in which it is placed. It will also, after a greater or less time, be carried to that part of the stream that has the greatest velocity. If the float, and the weight attached to it, have similar figures, and are of equal size, it will be obvious that the velocity assumed by it will be the mean of that of the two horizontal layers of the stream in which they lie. As this condition may be always conveniently adopted in practice, there is no need of investigating the correction that would be due to any difference in these respects. If the distance between the two bodies be small, their velocity may, without sensible error, be considered as that of the surface of the stream.

The float may have the form of a hollow cylinder, as for instance, a piece of a reed, closed at one end. This may be ballasted

by shot or other small weights, until no more than a small portion of it be elevated above the surface.

If the length of the tube bear a considerable proportion to the depth of the stream, it will float in a position considerably inclined to the vertical, in consequence of the difference of the velocity with which the fluid moves at different depths; and in all cases, there will be a greater or less inclination. An investigation of the nature and direction of the forces that act, may enable us to determine the velocity at the surface when the velocity of the reed and its inclination to the horizon are given. Such an investigation may be found in Venturoli, Vol. II. p. 218. It is, however, better not to use this instrument, except when it floats nearly vertically, in which case the error is unimportant, and therefore needs no correction.

In streams of still larger size, a part is chosen in which the water flows for a distance of 1 or 200 yards, without eddies. The area is measured as above, and the mean of several measurements is to be preferred to a single one. The sum of the length of the lower sides of the triangles and quadrilateral figures, is taken as the length of the curved section of the bed. The area divided by this, gives the quantity R, in (454), or the hydraulic mean depth. The slope of the surface, or I, of the same formula, is next obtained by levelling, taking the difference in the altitude of the surface of the stream at two points, whose distance is ascertained by measurement. These two quantities being given, the mean velocity is deduced by means of the formulæ, (462) or (463), and this multiplied by the area, gives the discharge in the unit of time.

An instrument used for a variety of other purposes, and called a Dynamometer, may also be applied to measure the velocity of a stream. The essential part of this instrument is a spring, the quantity of whose contraction, under pressures of known intensities, has been determined by experiment. If a plane surface be attached to such a spring, by a cord or other convenient method, the action of the stream will compress the spring, and the amount of compression will measure the action of the water, which will be given by the index of the instrument, in some conventional unit of weight. This action may also be considered as represented by the weight of a prism of the fluid, whose area is equal to the area of the surface acted upon, and whose altitude is that through which a heavy body must have fallen to acquire the velocity. If then the number of units of some given cubic measure of water that is equivalent to the weight to be calculated, and divided by the area of the plane, the quotient is that height; from this the velocity may be calculated by the usual formula.

This principle is not absolutely true, as will be seen when we treat of the percussion and resistance of liquids. It is, however, sufficiently near the truth for all practical purposes.

One other method of gauging a stream, remains to be mentioned. It is applicable to the case where it is traversed by a dam whose upper surface is horizontal, and over which the water discharges itself in a sheet, forming a water-fall. The velocity with which a fluid passing through such an aperture as the upper surface of the dam would represent, is discharged, is, according to the reasonings and experiments of Du Buat,

$$v = 0.58\, b \sqrt{h^3},$$

in which expression b is the breadth of the sheet of water, and h its depth, both expressed in English feet. The formula for the French metre, is

$$v = 0.1895\, b \sqrt{h^3}. \qquad (464)$$

These formulæ comprise the last case of the motion of liquids recapitulated in § 397.

CHAPTER VII.

Of Canals.

420. Canals are open artificial channels formed for the conveyance of water. They may be used for the purposes of navigation; for the supply of those intended for navigation, in which case they are called Feeders; for the supply of cities; in the draining of morasses, and for irrigating land for agricultural purposes. Canals differ in character from rivers; the latter have, by the long action of antagonist forces, formed beds, that generally speaking occupy the lowest levels in vallies, and follow the lines of greatest slope. The former have some conventional slope suited to their object, and which is usually uniform, or are absolutely level; they follow this prescribed line along the sides of hills, or through their masses, and are often carried at a considerable height over streams, ravines, and even broad vallies. Rivers rise from humble origin, but uniting with others as they proceed, and receiving the discharge from lateral vallies, swell in bulk in spite of the causes of waste that affect them equally with canals. Canals, on the other hand, maintain an uniform section throughout their course, and their volume diminishes in consequence of those causes; while if collateral supplies are brought in, they need be no more than equal to the restoration of what has been wasted: if these collateral supplies exceed this amount, means must be provided to get rid of the excess.

421. Canals take their rise in reservoirs, either in the form of natural streams and lakes, or of artificial basins, that collect the surface water. The fluid contained in these has usually a velocity less than that required in the canal; and even if it be a rapid stream, whence the canal proceeds, there will be a difference in the directions of the two currents. Hence the water in the canal will not at once assume the required velocity; and if the bed have a constant section until it unite with the reservoir, the area of the stream will diminish as it recedes from the reservoir, until the constant velocity adapted to the slope and circumstances of the channel be attained. This diminution of area can only take place by a diminution in the depth of the water in the canal. Hence at the origin of canals, whose area does not vary, a fall will be formed, by which the area of the stream will be diminished. If then it be important that the

canal shall be filled, it must spread out as it approaches the reservoir. If we suppose it to be influenced by the same causes that affect the vena contracta, and that the channel has a rectangular section of uniform depth, the plan of the channel would be formed on each side by a logarithmic curve, whose axis and greater and less ordinates have the proportion of 5 : 6.25 : 4; the second number being half the breadth of the channel at the reservoir; and the third half its uniform breadth. If the figure were investigated on the principle that the velocity of water moving in a channel varies inversely as its area, it would be found that the proper figure of the banks would be a parabola, whose vertex is at the point where the breadth of the channel becomes constant. Neither of these methods of investigation is free from objection; but it is evident from experiment and observation, that in order that water shall enter into a channel without forming a fall, or that it shall completely fill it, the channel must, at the reservoir, have a width greater than the breadth of the uniform section that it has at other points; and this increase of breadth should take place in the form of a curve convex towards the axis of the canal.

Such a form is to be found in nature when streams take their rise in lakes, or other similar reservoirs. If a canal be formed in soft earth, it will gradually wear away the earth until it assume such a form; but in solid rock such a shape cannot be spontaneously assumed. Even in canals made in soft earth, it is better to give the requisite shape artificially, than to wait for the slow action of the water.

422. The shape of the section of a canal is usually a trapezium, two of whose sides are parallel and horizontal; the other two equally inclined to the horizon. This inclination will depend upon the nature of the soil in which the canal is formed, being least in tenacious earth, and greatest in loose soils. No soil will maintain itself when the base of the slope is less than one and a third times its height, or in the proportion of four to three. While in loose soils the slope must be at least as great as in the proportion of two to one; or the base twice as great as the height. The force that acts upon the bank is the pressure of the water, and this is partly exerted to push the bank aside, and partly to overturn it by a rotary motion. In banks of earth, where the material has little cohesive strength, the former effort is most likely to be injurious; while in masses of masonry, the latter is the more important.

To investigate the proper thickness of a bank, whose height and slope and the material of which it is composed are given:

Let ABCD be the section of a prismatic mound of earth, pressed by water on the side AB; let the slopes on each side be equal, and suppose the bank itself to be composed of materials of infinite cohesive force, but capable of being moved horizontally in the direction AD, by a force equal to the friction among its particles.

C

Let AB, the length of the face	$= a$;
BAE, the base of the slope,	$= b$;
ABE, the vertical height,	$= h$;
the angle of the slope, ABC,	$= i$;
AD, the thickness of the top of the bank	$= x$;
the density of the material, that of water being unity,	$= D$;
the co-efficient of the friction	$= f$;
the thickness of the bank at bottom will be	$= x + 2b$;

The pressure on the line AB will be, § 331,

$$\tfrac{1}{2} a h;$$

this force acts at the centre of pressure in a direction perpendicular to the face of the bank. If then it be resolved into two components, one in a horizontal, the other in the vertical direction, the former only will tend to thrust the bank from its plane; the latter will add to the weight that presses on the base, will increase the friction, and consequently add to the stability.

The first of these components will be, § 13,

$$\tfrac{1}{2} ah \sin. i;$$

the second,

$$\tfrac{1}{2} ah \cos. i;$$

but as

$$\sin. i = \frac{h}{a}, \text{ and } \cos. i = \frac{b}{a},$$

these forces become respectively

$$\tfrac{1}{2} h^2, \text{ and } \tfrac{1}{2} h b;$$

of which that represented by $\frac{1}{2} h^2$, tends to thrust the bank from its place.

This force is resisted by the friction on the base, which is a function of the whole pressure, and this is made up of the weight of the bank, and the force $\frac{1}{2} h b$ derived from the liquid pressure acting downwards.

The weight of the bank is found by multiplying its area by its density, and is

$$(x + b) h D;$$

we therefore have, as the condition of equilibrium,

$$\tfrac{1}{2}h^2=\tfrac{1}{2}fhb+fhD(x+b); \qquad (465)$$

whence we obtain for the value of x,

$$x=\frac{\tfrac{1}{2}h-\tfrac{1}{2}fb-fbD}{fD}. \qquad (466)$$

When the bank is triangular, $x=0$, and the base of the slope becomes

$$b=\frac{h}{f+2Df}; \qquad (467)$$

to find the co-efficient f, we have

$$f=\frac{h}{b+2Db}. \qquad (468)$$

The mean density of earthy substances being about 2, we have in the case where $2b=3h$,

$$f=\tfrac{2}{15}=0.133;$$

and when $b=2h$,

$$f=\tfrac{1}{10};$$

and, as the pressure at the surface is evanescent, and a triangular mound would of course resist it, these values may be applied in other investigations.

In practice, however, the water will enter into the pores of the earth, and thus the surface exposed to pressure will be much increased.

To allow for this, let the co-efficients f be reduced to one half, or in firm earth to 0.06, and in loose earth, to 0.05. We shall then have, in the latter case, in a bank of triangular section,

$$b=4h;$$

and the whole base of the bank,

$$2b=8h. \qquad (469)$$

When the mass of the bank is given, as that part of the liquid pressure which tends to thrust the bank aside, diminishes as the inclination of the face on which it acts increases; while on the opposite side, loose earth supports itself at a slope whose base is to the height as 2 : 1: we may infer, that the maximum of strength will be attained when the slope on which the water presses has for its base

$$b=6h;$$

and when the base, b', of the opposite side, is

$$b'=2h.$$

In loose earth, when the bank has a trapezium for its section, and $b=2h$, we obtain for the value of x, from (466),

$$x=2.5(h); \qquad (470)$$

in firm earth, where $b=1\frac{1}{2}h$,

$$x=2.3(h). \qquad (471)$$

These give the thickness of the bank at the surface of the water; but as it is usual to make the bank of a canal one foot higher than the level of the water it contains, we would have for the thickness at its upper surface, in the case of loose earth, in feet,

$$x=2.5(h)-4, \qquad (472)$$

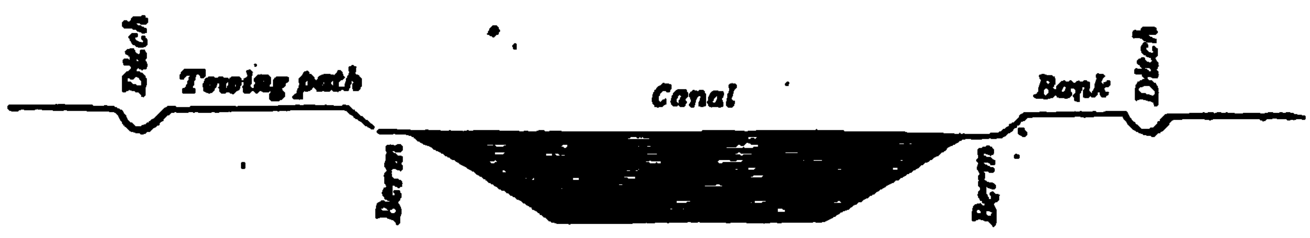
and in firm earth,

$$x=2.3(h)-3. \qquad (473)$$

423. A canal is usually confined between a bank on one side, whose dimensions may be determined on the foregoing principles, and a towing path, the breadth of whose upper surface must be sufficient for a road, on which the animals employed in draught may easily pass each other. If, then, the dimensions deduced above be not sufficient for this last object, the breadth of the upper surface of the towing path must be increased to at least nine feet. For the other bank, the usual rule is, to make its breadth at top equal to the height measured from the bottom of the canal; but in this case there should be a berm of from 1 to $1\frac{1}{2}$ feet at the level of the water, by which the thickness of the bank at the water's edge will, in usual cases, nearly coincide with our formulæ, (470) or (471, and which will have the advantage of preventing the wash of the banks from falling into the canal.

To prevent the entrance of rain water, a ditch called the counter ditch is formed on the outside of each of the banks. This is particularly necessary in side lying-ground, where the rain may produce injurious effects.

The profile of a well-constructed canal will therefore present the following figure:



And when the breadth and depth are given, the relations between the depth of excavation and the height of embankment, that will just suffice to give the proper form, is a simple geometric problem.

424. Canals are applied to the purposes of inland navigation in several cases:

(1.) As has been already stated, they may be employed to pass obstacles upon rivers, that are in other respects navigable, or may be constructed parallel to a stream whence they derive their supply of water.

(2). They may be made to communicate between two navigations of equal levels, drawing their supply from both, or between two of different heights, drawing their supply from the higher.

(3). They may form a communication between two navigations, passing over ground higher than either. This is at present the most usual case of canal navigation. Such canals are said to have a summit level, or to be *à point de partage*.

425. The dimensions of navigable canals will depend upon the section of the vessels intended to navigate them. They must in the first place be wide enough to permit two vessels to pass each other with freedom; for this purpose the breadth at bottom is usually made twice as great as the breadth of beam of the vessels; in the second place, the depth is usually made one foot more than their draught of water. This is done for two reasons: first, because it is found that vessels are much more impeded, when the channel in which they move is shallow, than when it is deeper; and secondly, in order to provide for any accidental defect of water, or for deposits of earth that may lodge in the canal. In either of these cases, the vessels would be exposed to take the ground, were there not an extra depth of water.

426. The bed of a canal must be absolutely level, or have no more slope than is sufficient to convey water to replace that which may be wasted. Hence, when the navigations it is to connect are of different levels; when constructed on the bank of a stream whose surface must of course slope; or when it surmounts a summit, it must be made in a series of channels at different levels, and means must be provided to pass vessels from one level to another. That which is most usually employed is called a Lock.

A lock is a four-sided chamber, contained between two parallel walls in the direction of its length, and by two gates or sluices, situated at the two extremities.

The bottom of a lock is a floor of wood, or masonry, on the same level with the bottom of the lower of the two reaches of the canal that it serves to connect. The gates rise to the height of the water in the upper reach, and the walls to the height of the banks of that reach. The lower gate reaches to the floor of the lock, the upper gate is usually established upon a transverse wall, the top of which is on a level with the bottom of the upper reach, and which is called the Breast Wall.

When the water in the lock is upon a level with the surface of the water in the higher reach, it is said to be full; when on a level with that in the lower reach of the canal, it is said to be empty. The gates are suspended from the walls in such a man-

ner as to close, when they undergo a pressure from above; they may be opened when the pressure on each side is equal.

A plan and section of a lock are represented beneath.

FIG. 2. FIG. 1.

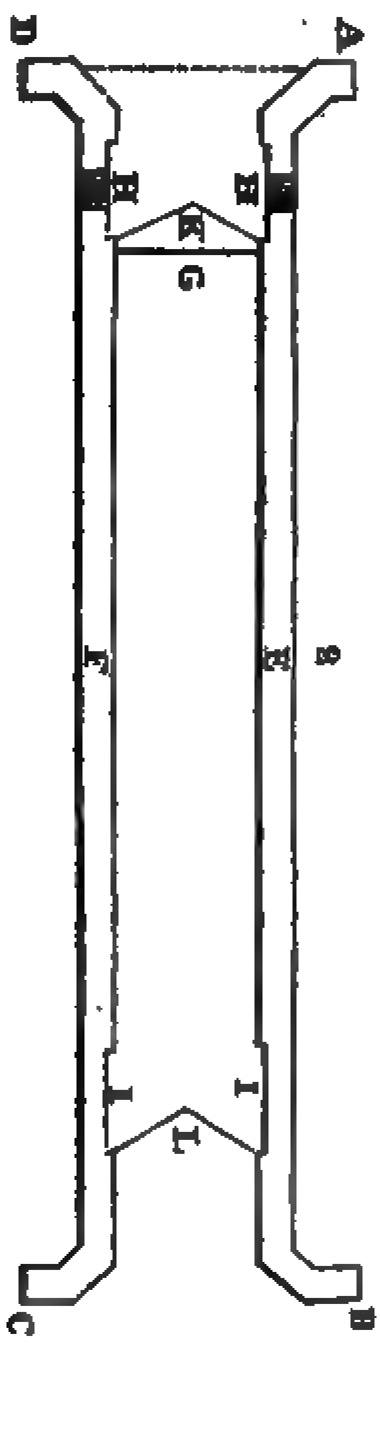


Fig. 1st, is a longitudinal section; fig. 2d, a horizontal plan of a lock.

A, B, C, D, are the walls of which E and F are the parts that enclose the chamber.

M, G, N, D, the breast wall on which the upper gate is supported.

HH, recesses in which the upper gate is received when open.

I I, similar recesses to receive the lower gate.

K, sill of the upper gate.

L, sill of the lower gate.

O, upper opening of the culverts that form a communication between the waters of the upper and lower levels of the canal, in order to fill the lock.

V, vault in the breast wall into which these culverts enter.

WW, level of the water in the upper level of the canal.

W'W', do " " lower do.

The operations of filling the lock from the upper level of the canal, and of emtying it, by a discharge into the lower level, may be effected by means of small gates, or wickets, sliding in grooves in the timbers of the gates.

To this method no objections apply in the lower gate; but in the upper, the water spouting from the top of the breast wall may injure the lock, or enter into the vessel contained in it, which may thus be destroyed, or sunk. Hence, passages called Culverts, have been formed in the masonry of the walls of the lock, to which wickets are adapted; these passes opening from the water in the upper level, are inclined in such a manner as to discharge themselves below the surface of the water in the lock, even when as much as possible is drawn off. In the Canal du Centre, in France, the culverts descend vertically, and discharge themselves into an open vault, formed by the thickness of the breast wall.

A lock has recently been invented in this country by Mr. Canvass White, in which the breast wall is omitted, and the upper gate is as tall as the lower one. In this form, the culverts in the walls become unnecessary, and the breast wall which is for many reasons, the weakest part of a lock, is suppressed. The bottom of the upper and lower levels, are united by a slope, rising from the sill of the upper gate.

427. When a vessel is to rise from the lower to the upper level of the canal, it may either find the lock full, or empty; in the former case, the surplus water must be discharged through the wickets, until it reach a common level on each side of the lower gate; in the latter, the water in the lock, and in the lower level are already at an equal height. The pressure being therefore equal on each side of the lower gate, it may be opened, and the boat drawn forward into the lock. The lower gates are next shut behind it, and the wickets or culverts that communicate with the water above opened. This will therefore pass into the lock, and as it finds no exit through the lower gate, will fill the lock; as the lock fills, the vessel, buoyant on the surface, rises; the filling of the lock, and rise of the vessel continue, until the water stand on

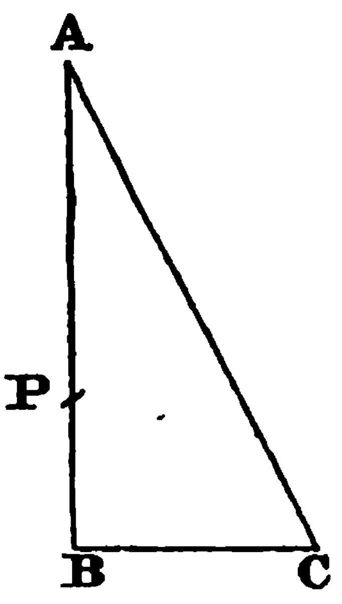
each side of the upper gate, at a common level. The pressure on each side of the upper gate being then equal, it may be opened, and the vessel drawn forward into the upper level of the canal.

If the canal be empty when a boat is to descend, it must be filled, or if full, kept so until the upper gate be opened, and the vessel admitted. The upper gate is then closed behind the vessel, and the water discharged from the lock, through the wickets of the lower gate, until the water within, and in the level below, reach the same height, when the lower gate may be opened, and the vessel drawn out.

The lock appears to have taken its origin from the accidental juxta-position of two sluices, in the action of which its important and valuable properties were discovered. It seems to have been first intentionally used in the Canal of Martizana, in Italy, about the eleventh century.

428. To determine the thickness of the longitudinal walls that confine a lock, when the depth of water, and the nature of the material is given, we must in the first place consider, that being built of masonry, the resistance to lateral thrust, is that of the friction of stone upon stone, at the joints, and of the cohesive force of the stone at other points; the former is aided by the cohesive force of the mortar, and these resistances being both great, the water will exert a more powerful influence to overturn the wall, than to move it laterally. As the pressure at the surface of the water is 0, and as the wall may be built in a vertical position, we may assume it, for the purpose of investigation, to have a section of the figure of a right angled triangle.

Let ABC, be a section of the wall; let the height AB, $=p$; the thickness BC, $=x$; the liquid pressure on the face AB, will be, § 417,



$$\tfrac{1}{2}p^2;$$

and it will act to overturn the wall at the point P, the centre of pressure, with a moment of rotation represented by

$$\tfrac{1}{2}p^2 \times \tfrac{1}{3}p = \tfrac{1}{6}p^3.$$

The resisting force will be the weight

$$\tfrac{1}{2}pxD;$$

and to find its moment of rotation it must be multiplied by the perpendicular distance of its direction from the point C, which is $\frac{2}{3}x$. This moment of rotation, then, will be

$$\tfrac{1}{2}px \times \tfrac{2}{3}x = \tfrac{1}{3}px^2;$$

and in the case of equilibrium between the two forces,

$$\tfrac{1}{6}p^3=\tfrac{1}{3}px^2; \qquad (474)$$

whence we obtain

$$x=\sqrt{\frac{p^2}{2D}}=\frac{p}{\sqrt{2D}}; \qquad (475)$$

if we take the density of the materials to be 2, we have

$$x=\tfrac{1}{2}p, \qquad (476)$$

or the thickness of the wall at bottom, should be equal to half its height.

In the construction of locks, the thickness of the walls at bottom is made equal to half the depth of the water they contain when full. But the top of the wall is higher than the top of the gates, which determines the highest level of the water, in order to prevent any accidental overflow; and the section of the wall is not triangular, but quadrilateral. The top must have a sufficient thickness to admit of the masonry being carried up in two faces of ashler, which are filled up within by irregular pieces, laid in water proof cement, and grouted, in order to prevent any filtration. This thickness cannot be less than from 3 to 4 feet. When the wall is of brick, it may be of less thickness at top, but must be covered with a coping strongly clamped. When the depth of water to be supported is less than 7 or 8 feet, the wall will have two parallel faces; at greater depths the outer face slopes, until it has a thickness at bottom of half the depth of water.

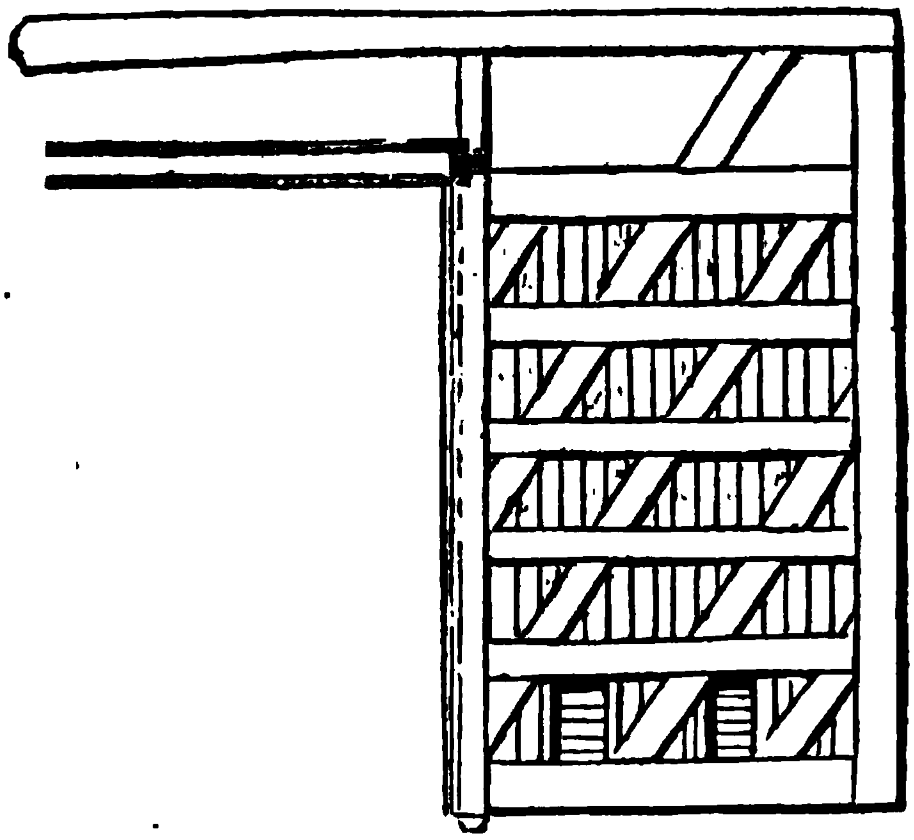
The walls of the lock are continued at its two ends, until they meet the banks of the canal; and as the latter is wider than the lock, these portions diverge, and are called the Wing Walls; those at the lower end of the canal decrease in height, until they meet the earthen bank. These walls are represented in the profile, and section at A D and BC.

These walls having less stress to undergo than those of the chamber of the lock, need not be thicker at bottom than one third of their own height.

429. The gates of locks are frames of timber, covered with plank; the lower gate reaches from the bottom of the lock; the upper from the top of the breast wall, to the level of the water in the upper reach of the canal.

As frames of a quadrilateral figure are liable to change their shape when suspended at one end, the frame must have diagonal braces, or the plank may be put on in a diagonal direction. The gate is suspended, by adapting a gudgeon to the bottom of one of its uprights, and passing an iron collar built into the wall over its upper end. This post must therefore project below the bottom of the gate, and a circular cavity is made in the bottom of the lock, to receive its gudgeon. This post, as well as that on the

opposite side of the gate, is sufficiently tall to rise above the top of the wall, in order to be framed into the lever that serves to open and shut the gate. This lever extends, when the gate is shut, over the top of the wall, and serves as a counterpoise to the gate. The form of the gates will be understood by reference to the figure.



The bottom of the lock is lower, within the space where the gates swing, than at any other place, and the gates when shut, rest against a sill. The walls also, are built in such a form as to admit the gate into their thickness, when open, in such manner that its face, and that of the wall, shall then be in the same plane. The outer side of the post on which the gate hangs is rounded, and the stones are cut, where it touches the wall, into a curved hollow, that permits its motion, and to which the retired part of the wall is a tangent.

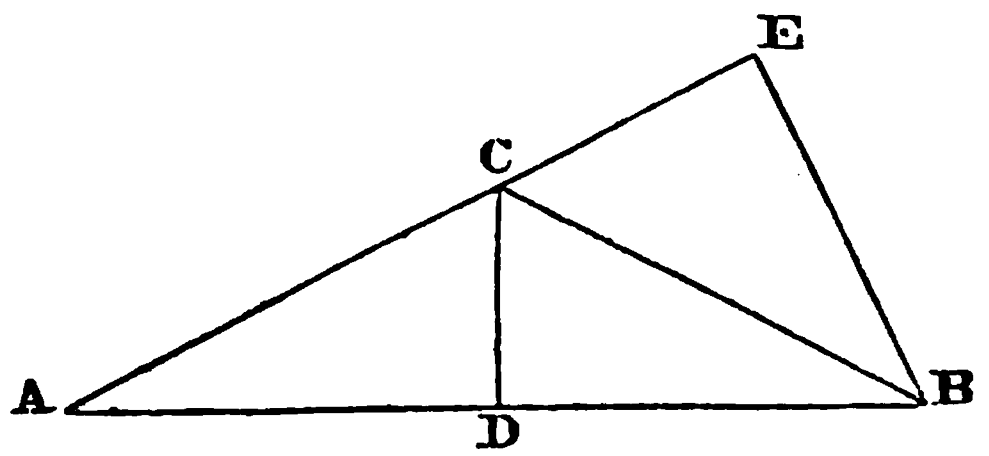
When the lock has a small breadth, say not exceeding four feet, both of the gates may be formed of a single leaf, and are, when shut, at right angles to the faces of the walls. When the breadth does not exceed six or seven feet, the upper gate may still be in a single leaf; but in this case the lower gate, and in all locks whose breadth exceeds seven feet, both gates must be formed of two equal, and similar leaves, suspended from the opposite sides of the lock. The reason of this is, that the pressure would then become too great to be borne by a single length of timber, supported at one end.

430. When lock gates are made of two leaves, they must be so placed as to afford each other mutual support. For this purpose, instead of shutting in such a manner as to lie in one plane,

the two leaves make an angle with each other, and the sill has the figure of an isosceles triangle, whose vertex is turned towards the upper level of the canal. This arrangement will be seen in the draught, on page 443.

It will be obvious, that if this angle be a right angle, the timbers of each leaf will receive the pressure of the other in the direction of their length, and will therefore afford the greatest mutual support; while if they lie in one plane, they will not give each other any support. But in the former case, the timbers will be longer, and have less strength to bear the liquid pressure that acts upon the leaf of which they are a part, than in the latter. Upon these principles, the proper angle at which the gates should meet, in order to possess a maximum of strength, may be investigated.

The pressure of water upon the leaf AC, when the depth is constant, will be proportioned to the length of AC, which we shall



call l; and the strength of the timbers which form the gate will be inversely proportioned to l; call the pressure on the unit of surface, P; and S, the respective strength of the material. We have in the case of equilibrium,

$$Pl=\frac{S}{l};$$

and

$$S=Pl^2.$$

The strength, then, in order to resist the pressure, must be a constant function of the square of the horizontal dimension of the leaf; or if we call the breadth of the lock $2a$, and the projection CD of the sill x, a constant function of (a^2+x^2).

The leaf AC, also derives strength from the other leaf BC: if we decompose this force into two parts, one of which is perpendicular, the other parallel to AC, and which may be represented by CE, EB; that represented by EB, will alone act to support the leaf AE. But this force will vary with the sine of the angle A, while x varies with the tangent; it will therefore be a constant

function of ax. The whole resistance which a gate of a given breadth will oppose to the pressure of the water, will decrease with the former of these quantities, and increase with the latter, of both of which it is a constant friction, or will be greatest when $ax-(a^2+x^2)$ is a maximum, or when

$$x=\frac{1}{2}a=a\sin.30^\circ.$$

Hence, the greatest strength will be attained when the projection CD of the sill is one fourth of the breadth of the lock, and the angle that the gate makes with the wall of the lock is 60°.

When the length of the horizontal timbers that form the leaves of a gate, is determined by means of the breadth, and the most advantageous projection thus investigated, their dimensions may be calculated by considering them as beams supported at each end, upon the principles of § 189.

431. The height which is given to locks, is the sum of two quantities: the depth of the water in the lower level of the canal, and the difference between the level of the two surfaces. The latter is called the Fall of the Lock. The fall that is most advantageous, will depend upon a variety of circumstances.

The nature of the ground, in some cases, prescribes the proper fall; but in the hands of a skilful engineer, the quantity of fall may, generally speaking, be settled upon other principles.

The fall of the locks of a given canal should be constant, so that the water discharged from one may just suffice to supply those below it. In this case, the successive levels should be each capable of containing one lock full of water, in addition to that which is necessary for the navigation, without overflow. The service of the canal would then be performed without interruption, and without waste. This rule, however, neglects the loss of water by evaporation and leakage: a better principle would be, that the fall of the several locks should decrease from the point at which the canal receives water, until a new supply be admitted, when they are again to be restored to their original height.

It is next to be remarked that locks of great fall expend a great quantity of water, and can therefore be used only when that is abundant.

In locks of small fall, boats require nearly as much time to pass each of them, as to pass one of greater height. Hence, when a given fall is distributed among several locks, instead of a single one, the delays are much increased.

The thickness of the walls increasing with the depth of water,

with which their height increases also, the cost of masonry wll increase with the square of the depth. This depth is not the fall of the lock, but is the sum of that quantity, and the constant depth of the canal. Thus the whole cost of the masonry of a set of locks, to overcome a given fall, will decrease with the increase of their fall, and diminution of their number, to a certain limit, easily determined in each particular case, and afterwards increase. In this investigation, other circumstances will come into view, such as the cost of wood and iron work; the expense of making and securing the foundations.

Locks are liable to danger from the filtration of water through the earth in which they are placed: this may, in some cases, be so great, that the bottom of the lock may be pressed upwards by a liquid pressure, due to the whole surface of the lock and the head of water in the upper level. To counteract this pressure, we have the weight of the lock itself, and that of the water it contains. In a high lock, when empty, the former force may preponderate, and the lock be forced upwards.

Taking all these circumstances into account, it has been laid down as a rule, that the fall of locks should not exceed ten, nor fall short of eight feet.

432. The length of a lock will depend upon that of the largest vessels that are to pass it, and will be such that a vessel can lie, without risk of touching, between the angle of the leaves of the lower gate and the breast wall.

The breadth of locks must, in like manner, be adapted to that of the vessels, and is usually made one foot greater than their breadth of beam. So also, as has been seen, the dimensions of the canal itself will depend upon that of the vessels that are to navigate it.

In determining the appropriate size of vessels, the nature of the adjacent navigable communications must be taken into account, as well as the expense of construction, and the character of the moving power employed upon them. As a general rule, the expense of construction increases even in a higher ratio than the dimensions of the canal; hence, when it is intended to connect two navigations already in existence, it should be calculated for the smallest vessels that usually ply upon them.

The moving power employed, is usually that of horses, walking upon the towing path of the canal; and a loss of power would ensue if the boats were smaller than is calculated for the draught of a single horse. It has been found, by long experience, that a horse readily drags twenty-five tons, in a boat weighing five tons, upon a canal, or thirty tons in all. Boats of this tonnage are, therefore, well-adapted to canal navigation, and may have a

length of about 60 feet, a breadth of beam of 8 feet, and a draught of water of 4 feet. On the other hand, it is to be considered, that the resistance to boats of similar figure, increases with the area of their midship frame, or in a ratio no higher than the square of their homologous dimensions; while the tonnage they carry increases with the cube: and that large vessels, on a canal, do not require a greater number of hands to navigate them than small ones. Two are, in all cases, sufficient; the one to steer the boat, the other to drive the horse.

433. In the earlier canals, it was a frequent custom to build locks in juxta-position, like steps of stairs, the upper gate of the one answering as the lower gate of the next, and so on. This disposition is faulty. (1.) Inasmuch as it causes a greater use of water. And (2.) Because it delays the navigation.

When the locks have a space between them, sufficient for two boats to pass each other, and all the locks are of the same height, a single lock full of water will suffice for the passage of a boat from the summit to the lowest level of the canal; and if the trade exactly alternate, the ascending boats will find the locks empty; and they will be filled, in readiness for the descending boats, by the operation of raising those that are ascending. When the locks are in juxta-position, the first descending boat passes through the system with but one lock full of water, but when another is to rise, it finds all the locks empty; and hence, there must be drawn from the higher level as many locks full of water as there are locks in the system. The second descending boat finds all the locks full, and the whole of them must be discharged in order to permit it to pass. It is therefore evident, that when the locks are combined in a system, they use a quantity of water as many times greater than when they are separate, as there are locks in the system. This waste will not be obviated wholly, by merely placing a basin between the locks, in which two boats may pass each other; but this basin must be large enough to receive and retain a lock full of water, and to permit that same quantity to be drawn from it without rendering the water too shallow. If the basin have not sufficient capacity, a part of the descending water will run to waste, and the boats may take the ground. If the circumstances of the ground bring the locks very near to each other, the breadth of the basin between them must be increased, until it comply with the above condition. In general, this condition may be attained by increasing the space between the locks; and it may be shown by calculation, that in a canal, the locks of which have eight feet fall, there should not be a less space than 200 yards between two contiguous locks.

In respect to the loss of time arising from the juxta-position of

locks: In an alternating trade, but one vessel can be in the system at a time, while if there be a space between the locks, there may be an ascending vessel in each lock, and a descending one in each intermediate basin. It is therefore evident that if n be the number of locks combined in a system, no more than $\frac{1}{n}$th part of the number of boats that would pass, were the locks isolated, can pass the system in a given time, when they are in juxta-position.

There are, however, local cases, in which a system of locks, in juxta-position, must be made use of, instead of an equal number of single ones with basins between them. In this case, the defects we have spoken of, may be obviated by making two complete systems by the side of each other. The expense of construction is thus increased, but it may be more than compensated by saving in other parts of the work. Thus, at Lockport, on the great Western Canal of the State of New-York, where the upper level of the canal lies on the top of a ridge of compact limestone, that descends rapidly to the lower level, it was found much cheaper to make two collateral systems of locks, than to excavate the space necessary for intermediate basins. In such a combination, one of the systems is used for the ascending, the other for the descending trade.

434. Where circumstances compel the use of deep locks, the waste of water that they occasion may be lessened, by making lateral basins to receive a part of the discharge, when a boat is descending, and to restore it to the lock when a boat is to ascend. One on this plan, is described by Belidor, but the necessity of such a form can rarely occur, and the cost of construction would be very great.

435. The supply of water that is needed for a canal, depends:

(1.) Upon the lockage, which will follow the law of the number of vessels that will probably pass. It is usual to assume, that a vessel will expend one lock full of water on each side of the summit. This will certainly be sufficient when the locks are not in juxta-position.

(2.) Upon the evaporation from the surface, from which must be deducted the quantity of rain that falls upon it. It has been found by experiments made upon the canal of Languedoc, that the annual quantity of evaporation is 32 inches. That is to say, that the allowance for this cause of waste must be equal to a parallelopiped of water, whose base is the whole surface of the water in the canal, and whose altitude is 32 inches. In most calculations, it has been customary to take this altitude at 36 inches.

(3.) Upon the leakage. This may take place through the banks of the canal, or through the gates of its locks. In the former case, it will depend upon the nature of the soil and the extent of the canal; in the latter, it will be a constant quantity, in canals whose gates have equal dimensions; for the leakage through the gate nearest the summit, will supply that which takes place in all the gate on the same side of the summit.

The leakage through the banks is greatest in new canals; for the banks, if undisturbed, are gradually tightened by their own pressure, and by the particles of earth which the water deposits in filtering through them. When the soil is porous, the canal may be lined with an earth retentive of water, or a portion of the middle of each bank may be built up with a soil of this character. If placed in the middle of the bank, a tough tenacious clay may answer the purpose. But when upon the inner surface, it is necessary that it should as well resist the action of running water, as the entrance of stagnant. For this latter purpose, a loamy earth, mingled with small pebbles, has been found to be the best. The operation of lining a bank with earth retentive of moisture, is called Puddling.

When such precautions have been taken, and the banks have become consolidated, it is the estimate of European engineers, that the leakage is twice as much as the evaporation, or amounts to six feet upon the surface of the canal, annually. In our country, this estimate has been found insufficient: it is, however, rather to be ascribed to a defect in the mechanical construction, than to any difference in the physical circumstances.

436. When a canal unites two navigable streams, and can derive its waters from one or both of them, or when it is parallel to a considerable stream, it may be considered that the supply of water cannot fail to be sufficient. But when it passes over a ridge or summit that divides waters falling in two directions, to obtain a sufficient supply will require considerable pains and precautions. As the canal will seek to traverse the ridge at its lowest points or gaps, a channel cut from the summit level, and continued along the side of the higher parts of the ridge, will intercept all the waters that flow upon the surface of these higher parts; and if it have a slope towards the canal, will convey them to its summit level. Such a channel is called a Feeder, and the supply it will furnish will be more certain should it intersect the course of streams. The quantity of water it will certainly furnish, may be ascertained by gauging the streams: the slope that must be given to it, will probably be determined by local circumstances; and the proper dimensions to convey the given quantity upon the given slope, may be ascertained upon the principles of § 411.

Even when no stream of magnitude is traversed by the feeder, it may, if the extent of ground higher than it be considerable, receive a sufficient quantity of surface water to feed a canal. This may be judged of, by knowing the extent of surface that the feeder will drain, and the quantity of rain that falls upon it. A particular investigation will be necessary in relation to the latter circumstance, for the quantity of rain is far more influenced by local causes, and particularly by difference of altitude, than the quantity evaporated.

437. In almost all climates, the quantity of water furnished by streams, or directly by rain, is various at different seasons. At one time in the year the supply will be in excess, at another in defect. To guard against the consequences of this inequality, reservoirs must be constructed. These are made by closing up the entrances of vallies, into which the feeders are conducted, and must lie at so high a level that the water will run from their bottom into the canal; for, when this is not the case, no more water can be drawn from them, than lies above the level, whence water will run to it. As reservoirs are liable to evaporation from their surface, they ought to be constructed in places that will admit of their containing a large quantity of water, with the smallest possible surface; their banks ought therefore to be steep, and capacity obtained by increasing their depth, rather than their breadth. The water should be drawn from their surface, in order that it may be free from earthy matter, which the liquid remaining at rest in the reservoir, will deposit.

The strength of the walls and banks, by which water is retained in reservoirs, follows the same law as that of the banks of canals, § 422, or the walls of locks, § 428, according to the material of which they are constructed.

438. It is in all cases of extreme importance, that none but clear water should be admitted into a canal. If this precaution be not observed, the canal will fill up, and lodgments will take place in the locks, that will prevent the working of the gates. Hence, reservoirs may fulfil an important purpose, by clarifying the water, even when unnecessary to equalize the supply. In the original execution of the canal of Languedoc, no precautions were taken to admit none but clear water; on the contrary, efforts were made to introduce into the canal, every stream whose course it intersected. The consequence was, that after a few years, it was under contemplation to abandon it, rather than incur the great expenditure demanded for maintaining it of a depth sufficient for navigation. Vauban, however, who was deputed for that especial purpose, pointed out the means of preventing this difficulty from

occurring again. For the same reason, it is necessary in the New-York canals to discharge the water from them, and excavate annually several inches of sediment. The banks are in consequence exposed to the action of the frost, and are rendered liable to give way on the re-admission of the water. These inconveniences may be obviated by certain simple precautions:

(1.) No water should ever be admitted into a canal, until it has remained in a reservoir long enough to discharge its impurities; and the water should, as far as possible, be admitted at but a few points in the higher levels, whence the rest are to be supplied.

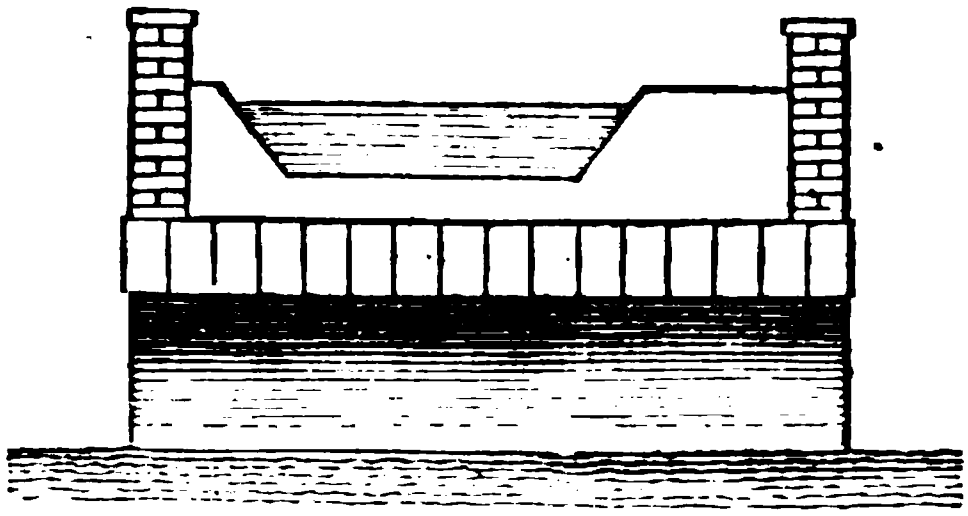
(2.) When a stream intersects the course of a canal, it is not to be admitted, but passed under it by a culvert, or over it by an aqueduct.

(3.) The rain, and surface waters of the country, that lie higher than the canal, must be intercepted by the counter ditch, and passed at proper places beneath the canal, to the lower country.

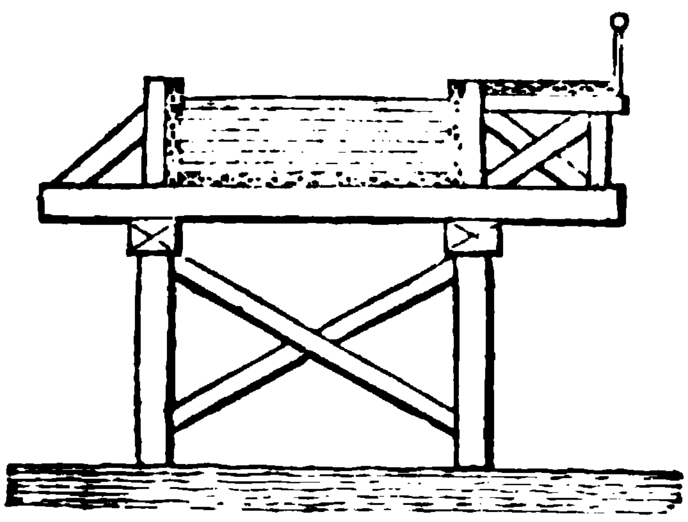
439. When water is to be passed beneath a canal, a Culvert must be constructed. This is an arched channel of masonry that is built beneath the bed of the canal, and reaches from one side of its embankment to the other. If the water to be conveyed, lie at about the same level with that in the canal, the culvert has the form of an inverted syphon, and acts upon the principle of a water pipe. The water may then be raised to nearly the same level at the end of the culvert, opposite to that at which it enters, and may resume its former channel. When used to convey water whose level is considerably lower than that in the canal, it may be considered as an open channel, the arch performing no other office than to support the embankment of the canal. The mere surface waters should be conveyed beneath the canal, at regular intervals, and by small culverts; but streams will require culverts adapted to their capacity; and it is generally better to unite a number of small culverts, than to make the arched passage of too large a size.

440. When the stream is large, or its valley wide, it becomes necessary to convey the canal across it by an aqueduct. An aqueduct is a bridge, that instead of carrying a road, contains the channel of a canal between its parapets. If built of masonry, it also contains the towing path and bank of the canal, and a sufficient mass of earth beneath its channel, to prevent filtration. The breadth of the channel is usually but half as great as in other parts of the canal, and therefore but one boat can pass the aqueduct at a time.

The following figure represents the section of an aqueduct of masonry.



In our country, aqueducts are frequently constructed, by forming a channel for the canal, of plank confined by frames of timber, as in the figure beneath. The towing path is also formed of wood.



In England, cast iron formed into plates, and united by screw bolts and nuts, is often used for aqueducts; they may be supported on pillars of stone or iron, and are perhaps the very best kind of aqueducts, except where the cost of the material much exceeds that of stone or wood.

The most remarkable aqueduct of this description, is upon the Ellesmere Canal, across the valley of Llangollen, in Wales.

The principles upon which the dimensions of the several parts of aqueducts, may be calculated, are the same with those of bridges, and of the banks of canals, or the walls of locks, and need not to be repeated.

441. Feeders or other sources of supply, may bring into a canal more water than can be used in lockage, or wasted by evaporation and leakage. This, running over the gates of the locks, would cause a current that must injure the banks, and if the levels be long, might swell and overflow them. To prevent the water accumulating, Waste Gates are constructed at intervals. These are built of masory, and have the shape of a triangular prism, the edge of which is at the level beyond which the water is not to be permitted to rise. If made in the towing path, a bridge is provided for the passage of the horses employed in draught.

These waste gates are placed where a natural channel has existed, by the continuation of which the surplus waters may be carried

off; or where such channels do not occur, an artificial channel is formed for the purpose.

442. When a country is mountainous, and the construction of canals would be attended with great and sudden changes of level, locks become too expensive to permit their application; and it thus happens that in many districts, where canals might be supplied with abundance of water, and the wants of commerce demand them, they are notwithstanding considered impracticable. In such cases modifications of the inclined plane, rendered self-acting, might no doubt be effected. But although often proposed, and in forms to which no reasonable objection can be applied, they have not yet been brought into successful action.

443. When water is scarce, canals are impracticable, or at least susceptible of conveying only a limited trade. Some of the modifications of the inclined plane use less water than locks, and might be advantageously employed in such cases.

Betancourt has proposed a lock, which vessels may pass without any expenditure of water whatever. A basin of area equal to the lock is built beside it, and communicates with it at bottom. In this basin is placed a water tight case that has nearly the same area with it, and is of the same density with water. A small force will therefore be sufficient to sink or raise it at pleasure. If the gates of the lock be closed, and this case be depressed, it forces water from the basin to the lock, until it reach the level of the upper reach of the canal, and a boat may be thus raised up. If a boat is to descend, the gates are again closed, and the case drawn upwards, the water in the lock then flows back into the lateral basin, and the boat floating upon it falls to the lower level. As the depth of water in the basin varies, a consequent variation will be demanded in the force that e resses the case. This is effected by a counterpoise, extremity of the arm of a bent level, in an arc of plane is vertical. The whole arrangement may be y the inspection of the figure on the succeeding p

A B C D E, are the walls enclosing the chamber of the lock H, and the lateral chamber I. F, is a wall separating the chamber of the lock, from the lateral basin. In this wall is the arched opening G, forming a communication between the chamber, H, and basin, I. L, represents the upper gate of the lock resting on the breast wall. K, a hollow plunger, by the elevation or depression of which the common level of the water in the basin and chamber, is raised or lowered. *m m*, a chain passing over the pully, N, and connecting the plunger, K, with the bent-lever, *m* O P, at the extremity, P, of which the counterpoise is situated.

This counterpoise moves in the quadrantal arc, *r* P *s*; and when the plunger is depressed to its greatest depth, presses vertically on the fulcrum, O, and has no action on the plunger; in other positions in the quadrant, its action, as has been demonstrated by Betancourt, increases in precisely the same ratio as that part of the weight of the plunger, which is not supported by the fluid pressure of the water in the basin.

CHAPTER VIII.

Of the Percussion and Resistance of Fluids.

444. When a surface is exposed to the action of a fluid, in motion, or when a surface in motion impinges against a fluid, if we have no regard to what occurs after impact, the circumstances may be considered as identical; that is to say, the action will, in the one case, depend upon the velocity with which the fluid strikes the surface; in the other, on the velocity with which the solid strikes the fluid. When both are in motion, the effect will obviously depend upon the difference between the two velocities, or, what is called the relative velocity, and either may be considered as being in motion with this velocity. Whichever of the two is in motion, or if both be in motion, we may consider the action as identical with a resistance, opposed by a plane at rest to a fluid moving with the relative velocity; and the term Resistance may be, in all cases, employed to denote the action.

The theoretic investigation of this problem is attended with great difficulty, inasmuch as the particles of the fluid interfere with each other's action, even before impact, and continue to act after impact, according to laws that it is impossible to ascertain.

445. If we suppose that the fluid particles strike in succession against the surface exposed to them, losing by their action so much of their velocity as is in the direction of a normal to the surface, and that they then cease to act, either upon the surface, or on the remaining particles, we have the hypothesis that is most commonly employed in this investigation, and which we shall now make use of.

Let a fluid, whose density is s, strike with a velocity v, at right angles against a plane surface whose area is A. Let h be the height due to the velocity v, dx the space through which the fluid passes in the time dt.

We have therefore from, (53)

$$v=\frac{dx}{dt}. \tag{477}$$

The quantity of fluid that strikes against the surface in the time dt, will be Adx, and its mass

$$A\,s\,dx; \tag{478}$$

and as it moves with the velocity v, its quantity of motion in dt will be

$$A\,s\,v\,dx; \tag{479}$$

and in the unit of time

$$\frac{A\,s\,v\,dx}{dt}; \tag{480}$$

substituting the value of $\frac{dx}{dt}$ from (477), we have for the resistance R,

$$R = A\,s\,v^2 = 2g\,A\,s\,h. \tag{481}$$

If the fluid be water, $s=1$, and

$$R = A\,v^2 = 2g\,A\,h. \tag{482}$$

In the case of direct impact then, the action of a liquid in motion against a surface, should be proportioned to the area of the surface, and the square of the velocity.

If the fluid strike against the plane at an angle of incidence i, the velocity must be resolved into two components, one of which is parallel, the other perpendicular to the plane; the latter will be represented by $v \sin. i$, and the formula (482) becomes

$$R = A\,(v \sin. i)^2 = A\,v^2 \sin.^2 i. \tag{483}$$

In oblique incidences then, the action of the fluid should be proportioned to the area of the surface, the square of the velocity, and the square of the sine of the angle of incidence.

To apply this theory to cases that may occur in practice:

Let the body be a prism whose section is an isosceles triangle; let the area of the rectangular base be A; the angle of the vertex of the triangle $2\,i$: The resistance on each of the sides is

$$\frac{1}{2}\,A\,v^2 \sin.^2 i,$$

and the whole resistance,

$$A\,v^2 \sin.^2 i. \tag{484}$$

If the triangle be right-angled, the resistance on its faces will be

$$\frac{A\,v^2}{2}. \tag{485}$$

It may also be shown, by the application of the calculus to the same theory, that the resistance to a half cylinder, is two thirds of that to the rectangle which forms its base; and that the resistance to a hemisphere is one half of that to a plane surface of the size of its great circle.

446. These investigations being of no practical value, we shall omit them, and proceed to the results of experiment, in the case to which the hypothesis bears the closest analogy, namely, that of a jet of fluid striking against a plane surface.

From the best experiments that have been made in this case, namely, those of Bossut, it has been concluded:

(1.) That under similar circumstances, the action of the fluid is nearly proportioned to the area of its section.

(2.) That all other circumstances being equal, the force is nearly proportioned to the square of the velocity.

(3.) That the force of the fluid, when the plane is oblique to the direction of the current, does not follow the law of the squares of the sines of the angles of inclination. At least at the angle of 60°, the resistance is always less than would be due to that ratio.

(4.) The absolute measure of the force with which the fluid strikes the plane directly, is not constant, or as given in (482)

$$2 A g h,$$

but varies according to the ratio that the surface bears to the section of the vein. If the surface be much greater than the section of the vein of fluid that strikes it, the above formula is true; but as they approach more nearly to equality, the force of impact becomes less, until, when they are about equal, the force may be represented by

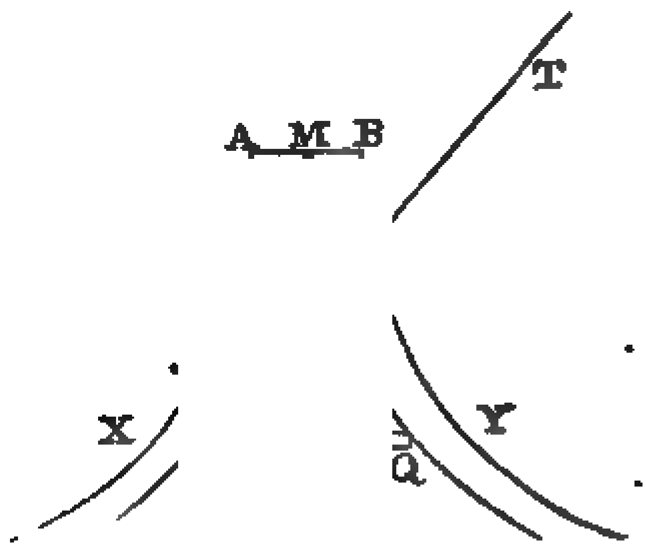
$$\frac{3}{4} A g h; \qquad (486)$$

or is no more than three eighths of that derived from the hypothesis.

When the phenomena with which these experiments are attended are carefully observed, they are found to be as follows:

The vein of fluid is enlarged as it approaches the surface, forming a conoid whose curvature is convex towards the axis, and varies with the greater or less size of the surface. The particles, after striking the surface, move off, if the surface be small, at a great angle of obliquity; and this angle lessens as the surface increases, until they move parallel to the surface, or as it were, slide along it. When this takes place, the resistance becomes equal to that deduced from the theory, and is no farther increased by an increase of the extent of the surface. But if the surface be surrounded by a ledge or rim, the resistance is increased beyond that deduced from the hypothesis; and in a concave surface, the resistance which ought, according to the hypothesis, be less than if it were plane, is also increased beyond that given by the formula (482).

These circumstances may be reduced to the test of analysis.



Let PQ be a surface exposed to the action of a jet of liquid, and let the curved lines, ALX, BY represent the boundaries of a section of the vein made by a plane passing through the axis MN, and let the vein be a solid, generated by the revolution of either of these similar curves, and assume the surface acted upon to have a circular section. Let M be the point where the vein commences to spread. We may conceive that the particles in motion divide themselves at the point M, leaving within them a conoid of stagnant fluid PMQ, and that the action upon the plane is transmitted to it by this conoid. We may also consider that the velocity in the current that surrounds PMQ, is constant, as there is no obvious reason for its change, and that it is equal to that the vein had before it began to spread.

This being premised, let the area of the vein at AMB $= a$; its constant velocity $= v$; and, referring the curve to the axis MN, let MG $= x$; GH $= g$; the arc MH $= s$, and HL, the breadth of the current at H $= z$; the radius of the osculating circle, at H $=$ R; the angle NPT of the obliquity with which the particles leave the surface $= \varphi$. As the velocity is conceived to be constant, the area of the current contained between the corresponding circular sections of the outer and inner cone will be constant also, and this area is formed at H, by the revolution of the line HL, which is a normal to both the curves. If HL be bisected in O, and OF $= y'$ be drawn perpendicular to the axis, this area will be

$$2\pi y' z.$$

If we take an elementary ring of the moving fluid, whose section is H l, its mass will be

$$2\pi g' z\, ds;$$

and its inner boundary, generated by the revolution of the line H h,

$$2\pi g\, ds.$$

Now by the principles of § 65, every element of the fluid contained in such a ring presses upon its inner boundary with a central force, represented by (85),

$$\frac{v^2}{R};$$

and this force multiplied by the mass, $2\pi y' z\, ds$, will give the moving force with which the fluid in the elementary ring presses upon its inner boundary, $2\pi y\, ds$. This product is

$$\frac{v^2}{R} \cdot 2\pi y' ds. \qquad (487)$$

This pressure may be represented by the weight of a cylinder of the fluid that has for the area of its base $2\pi y\, ds$, and for its altitude P, or by

$$2\pi\, g\, P\, y\, ds;$$

and

$$2\pi g\, P\, y\, ds = \frac{v^2}{R} 2\pi y' z\, ds;$$

whence we obtain for the value of P,

$$P = \frac{v^2 z}{g\, R} \cdot \frac{y'}{y}; \qquad (488)$$

and from the general distinctive property of fluids, every point of the surface, PQ, will be pressed by a column whose altitude is P.

If now in the preceding equation we substitute, for z and R, their respective values,

$$z = \frac{A}{2\pi y'}, \quad R = \frac{-dy}{d \cdot \frac{dz}{ds}},$$

we obtain

$$2\pi g\, P\, y\, dy = -A\, v^2 d \cdot \frac{dx}{ds}; \qquad (489)$$

and integrating

$$\pi g\, P\, y^2 = \text{Const.} - A v^2 \frac{dx}{ds}. \qquad (490)$$

When $y=0$, we have $\frac{dx}{ds}=1$, and when $y=PN$, we have $\frac{dx}{ds}=\surd$ $\sin. \varphi$; we therefore have for the integral,

$$\pi g\, .\, P \times PN^2 = A v^2 (1 - \sin. \varphi); \qquad (491)$$

but the first member of this equation is obviously the value of the resistance, for it is the weight of a cylinder of the fluid whose base is the circle PN, and whose altitude is P; the resistance, therefore, is

$$A v^2 (1 - \sin. \varphi). \qquad (492)$$

It is evident from this, that as the angle φ decreases, the resistance should increase until $\varphi=90°$, or when the fluid moves along the plate; in this case it becomes

$$A v^2,$$

according to the hypothesis, and these results are in conformity with the experiments.

If a ledge be formed around the surface PQ, the fluid cannot escape on the side opposite to that on which it impinges, or move along the surface; but must be thrown off in a direction making the angle φ on the same side of PQ with the axis MN; in this case $\frac{dx}{ds}$ becomes negative, and when $y=PN$, $\frac{dx}{ds}=-\sin.\ \varphi$; the expression for the resistance therefore becomes

$$Av^2(1+\sin.\ \varphi), \qquad (493)$$

and when $\varphi=90°$,

$$2Av^2. \qquad (494)$$

We may thus explain the very great difference in the resistance which a body of the form of a portion of a hollow sphere meets, when moving in a fluid, with its different surfaces opposed to it. This difference, it will appear from the theory, may be in a relation as great as four to one; the concave surface being resisted four times as much as the convex; and in practice, the difference is even greater. Upon this principle the parachute is applied to balloons.

447. When a surface, instead of being struck by a vein of fluid, is immersed in a mass of that description, and has a relative velocity in respect to it, growing out either of its own motion, or that of the fluid, or a combination of both; a similar conoid of stagnant water will be formed in front of it, and a similar diminution of the resistance will take place. We can only ascertain the quantity of this diminution by experiment, which shows that when the area of the stream is great in proportion to that of the surface, the resistance may be represented by

$$\frac{A v^2}{2}=A gh; \qquad (495)$$

or is no more than the half of that pointed out by the hypothesis.

If, however, the channel be narrow, and the surface fills up a large portion of it, the resistance augments. The experiments of Du Buat have given the following results;

Let the resistance in a channel of indefinitely large size be 10000, and M the ratio between the area of the channel and that of the obstacle, the resistance in the channel will be

$$\frac{84600}{M+2}. \qquad (496)$$

If, then, the obstacle fill up half the channel, the resistance is more than double what it is in an indefinitely large channel, and we may consider the channel as large enough to get rid of this cause of increased resistance, when its area exceeds $6\frac{1}{4}$ times that of the obstacle.

448. When the surface acted upon, is under the circumstances of a floating body, one part being immersed in the fluid, the other floating above it, the level of the water is raised on the side on which the fluid acts, and a depression takes place on the opposite side.

The fluid striking against a plane surface with the velocity, v, will tend to rise in front of it, to the height due to this velocity, or to (60)

$$h=\frac{v^2}{2g}.$$

In like manner, a depression will take place behind a solid body, and if the surface be a plane parallel to the anterior surface, the whole difference of level will be,

$$\frac{v^2}{g}.$$

Upon an elementary rectangle, whose constant breadth $= b$, and depth $= dx$, the action will be due in part to the velocity with which it moves through the fluid, and partly to a pressure due to the depth, x, and may therefore be represented (495) by

$$\tfrac{1}{2}\, b\, d\, x\, (v-\sqrt{2\, g\, x})^2. \qquad (497)$$

Integrating, we have, for the additional pressure growing out of the rise of the fluid,

$$\frac{bv^4}{24\, g}. \qquad (498)$$

An equal resistance will grow out of the depression behind, so that the whole increase growing out of this cause, will be

$$\tfrac{1}{12}\cdot\frac{bv^4}{g}. \qquad (499)$$

It will be obvious, however, that the fluid cannot rise to the height due to the whole velocity, and therefore although the resistance growing out of this cause, probably varies with the fourth power of the velocity, it has a co-efficient considerably less than is determined by the above investigation.

449. We may therefore infer that the resistance of water to a body floating upon its surface, is composed of two separate forces, the one due to the impulse of the fluid; the other to the wave raised before, and the depression that takes place behind it. The first varies with the square, the second with the fourth power of the velocities. The first, however, is alone sensible at moderate

velocities, but the second becomes the most intense at great velocities, and must finally cause a limit beyond which the speed of a body moving at the surface of a liquid cannot be carried. No such resistance affects bodies wholly immersed in a liquid, and hence, as the limit of speed depends upon the square of the velocity, it is not as soon attained as in the former case.

450. Our investigations appear to show that the resistance to surfaces inclined to the direction at which the fluid strikes them, varies with the square of the sine of the angle of incidence. This is found to be far from true, in the case of bodies moving in masses of fluids, when submitted to the test of experiment. The theory of Juan, makes this resistance to vary with the sine simply; this is however equally, or even more faulty, except when the angle is great; and even in this case it gives a result below the truth.

If the inclination of the surface to the direction of the fluid, do not exceed 30°, or the angle of incidence is between 60° and 90°, the formula (483) corresponds nearly with the truth; at inclinations from 30°, to 60°, the following formula deduced by Bossut, from experiment, will give results nearly accurate.

$$R=\frac{av^2}{2}\sin.i^2+0.003\,(90°-i)^{3.25}. \qquad (500)$$

At still greater obliquities, the following formula of Romme, gives a nearer approximation.

$$R=\frac{av^2}{2}\,.\,30\,.\,\frac{2+\sin.i^2}{180°-i}\,. \qquad (501)$$

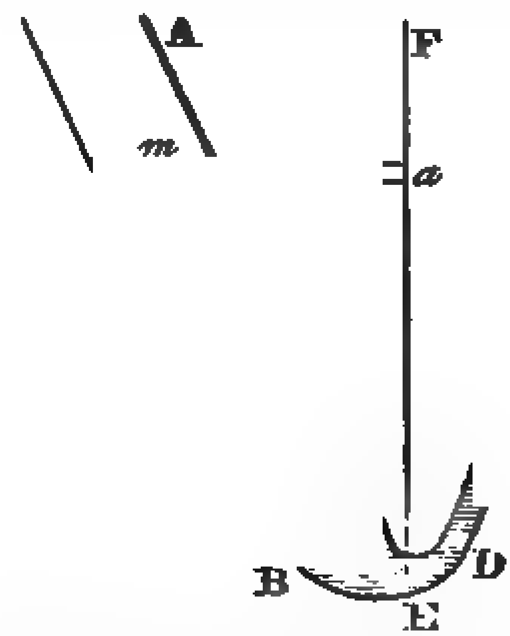
CHAPTER IX.

Of the Motion of Waves.

451. When a pressure is applied to any portion of the surface of a liquid, the column pressed is shortened or diminished in depth; the excess will enter into the surrounding columns, and the pressure will, from the nature of fluids, be propagated to them. They will in consequence rise above their original level. But not being sustained by a hydraulic pressure, they will again fall, and thus acquiring a velocity due to their height, will descend below the level, until that velocity is overcome by the action of the adjacent columns. These will, in consequence, receive a similar motion, which they will in turn propagate. Thus a series of ridges and intermediate cavities, will be formed upon the surface, and appear to propagate themselves in circles, from the column to which the original impulse was given. This motion that appears to take place on the surface, is not a progressive motion of the mass, for the particles of liquid that pass from one column to another, return again to that to which they originally belonged; and even this takes place below the surface. The propagation of motion, therefore, that takes place at the surface, is the communication of a tendency to oscillate, and cannot, so long as these oscillations are unimpeded, give a progressive motion to the particles at the surface.

452. The phenomena of the motions of waves have been compared to those of a fluid oscillating in a bent tube, and although the analogy is not complete, yet there are so many points of coincidence, as to authorize us to adopt this phenomenon as the foundation of our theory.

Let AB, CD, represent two branches of a bent tube, equally inclined to the horizon. Let the section of the branch AB$=m$, the



section of the branch CD$=n$. Let the vertical height EF of the tube $=p$, and the height of the level of the liquid in the branch AB, above the common level when at rest $= a$.

The branches being equally inclined, the spaces, in the direction of the axes of the respective branches, through which the liquid oscillates, will be proportioned to the vertical altitude of these spaces. Call the vertical height through which the liquid oscillates in the branch AB, x. Then as the liquid that is depressed in the one branch, enters into the other, the quantity that rises above the original level in each will be equal; the altitudes in the respective tubes, will be inversely as their respective areas; and the vertical height of the oscillation in CD, will be

$$\frac{mx}{n}.$$

453. To apply this to the case of waves diverging in circles from a point. They may be considered as formed by the vibrations of liquids, in a series of concentric cylindric tubes; the intervals between which are infinitely small. Their respective areas, are therefore proportioned to the circumferences of the circles that form the sections of the cylindric tubes, and as these are proportioned to their diameters, the heights to which a circular system of waves rises, will decrease in the inverse ratio of their distances from the point of original action, or in an arithmetic progression.

The force by which the fluid is caused to oscillate, will be the difference of the pressures of the columns in the two branches; this will be measured, § 331, by multiplying the area, m, by the

height a, supposing the density of the liquid $=1$, or will be ma. The column of fluid, whose pressure in the opposite branch is equal, will be measured by the product of the area, n, into a height which is to a, inversely as the respective areas, or by

$$\frac{amn}{n} = am ;$$

and the motion acquired by the column in AB, will continue until the pressure in CD become equal to the original force in AB, or until the column in CD becomes

$$2\,am ;$$

hence, the space through which the oscillation in CD is performed, becomes equal to

$$\frac{2\,am}{n},$$

and

$$z = 2\,a .$$

The column in CD will now preponderate, and by a similar course of reasoning may be shown to descend through $\frac{2na}{m}$, and to elevate the column in AB through $2\,a$. Thus, if we abstract from friction, the force exerted in each oscillation will remain constant; the oscillations will therefore be constant in extent, and the spaces described will be proportioned to the forces; whence we may at once infer that the oscillations are isochronous.

The times of these isochronous oscillations will depend upon the space through which the motion is performed, and the intensity of the moving force; and as the latter is equal in both directions of the oscillation, the circumstances will be the same as if both branches of the tube had equal areas.

Let the branches be of equal diameters, and let m, be the common area of the two branches, and a, the original elevation or depression from the common level. The difference between the masses contained in the two branches of the tube, will be $2\,am$, and their sum will be lm; l, being the whole length of the column in both branches, or of the axis of the tube. Applying the principle of D'Alembert, it may be shown as in the investigation of the principle of Atwood's machine, § 95, that the accelerating force at the beginning of an oscillation is represented by the difference of the masses divided by their sum, or by

$$\frac{2\,am}{lm};$$

which is equal to $\frac{2\,a}{l}$. But this force will be variable, because the difference of level in the two branches of the tube is continually changing. When the liquid in one branch has descended through the distance x, it will have risen an equal quan-

tity in the other branch, and the accelerating force f, will have become

$$f=\frac{2a-2x}{l};$$

and, as in the formula, (104a),

$$dt=\frac{2a-2x}{l}\,gdt; \qquad (502)$$

and from the general formula, (53),

$$dt=\frac{dx}{v};$$

substituting this value of dt, in (502), we have

$$vdv=\frac{g}{l}\,(2\,adx-2\,xdx).$$

Integrating, and rejecting the constant quantity, because when $a=0$, $v=0$,

$$\frac{v^2}{2}=\frac{g}{l}\,(2\,ax-x^2);$$

whence

$$v=\sqrt{\left[\frac{2g}{l}\,(2\,ax-x^2)\right]}; \qquad (503)$$

substituting this value of v, in the general expression, (53), we have

$$dt=\frac{dx}{v},$$

whence

$$dt=\frac{dx}{\sqrt{\left[\frac{2g}{l}\,(2\,ax-x^2)\right]}}; \qquad (504)$$

integrating, and rejecting the arbitrary constant, because when $x=0$, $t=0$, we have when $x=a$,

$$t=\tfrac{1}{2}\,\pi\sqrt{\left(\frac{\tfrac{1}{2}l}{g}\right)}, \qquad (505)$$

which is the time of half an oscillation. The time of a complete oscillation is therefore

$$t=\pi\sqrt{\left(\frac{\tfrac{1}{2}l}{g}\right)}. \qquad (506)$$

This is by (286) the time in which a cycloidal pendulum, whose length is half l, would perform its vibrations.

454. To apply this to the case of waves. A wave is continually oscillating between its highest and lowest points, and the motion being considered analogous to that of a fluid in a bent tube, the time of an undulation will be that of the oscillation of a

pendulum, whose length is half the distance between the highest and lowest points. The wave will return to its original height, or will appear to pass over the distance between two contiguous elevations or depressions in twice this time, or in the time of a single vibration of a pendulum, whose length is four times as great as that whose oscillations correspond with those of the waves. But if we abstract from the elevation of the wave, the length of such a pendulum is the distance between two contiguous elevations, or which is called the Breadth of the wave.

If we call this breadth b, the time T, of a wave running its breadth, may be represented by

$$T = \pi \sqrt{\frac{b}{g}} = \pi \sqrt{\frac{2\,l}{g}}. \qquad (507a)$$

From this we obtain for the value of the mean velocity with which the wave is propagated, by substituting in the general formula,

$$v = \frac{s}{t},$$

the value of $s = 2\ l$, and the foregoing value of T,

$$v = \frac{\sqrt{2\,lg}}{\pi}.$$

455. When a series of waves are proceeding from a centre, and meet a vertical obstacle, they are reflected; for the effect of the obstacle will be the same as if a new impulse were given to the column, causing an elevation or depression, equal to that they actually have; and this will be propagated like the original impulse, but in a contrary direction. The waves moving in circular arcs, the reflection takes place in similar circular arcs; and thus, the series of reflected waves will proceed exactly as if it came from a centre as far distant behind the vertical obstacle, as the original centre is in front of it. If the series of waves flow parallel, and of equal height, which, as we shall presently see, may occur, the reflected waves will diminish in height, as if they proceeded from behind the obstacle, and the joint elevation of the wave in immediate contact with the obstacle will be higher than any other.

456. The original, and reflected wav
attended with a progressive motion, wil
other's progress; but where the elevatio
correspond, the elevation is the sum of
depressions correspond, the resulting d
of the two depressions. Where a depre
sponds with an equal elevation in another, the surface of the liquid will not change its level. And in two series of circular waves, there will be certain points symmetrically arranged in curves,

as might be readily shown by a diagram, in which these effects will counteract each other; and thus, although different series of waves do not interfere with each other's propagation, they do still neutralize each other's effects, at particular points. This action is styled Interference.

457. If the obstacle against which waves strike, have a vertical opening in it, of small horizontal breadth, the oscillating columns that reach it, will act there, as an impulse originally exerted at that point would have done; and hence, a new series of waves will appear to proceed from the orifice. If the breadth of the orifice be increased, new series of waves will appear to proceed from it, but they will no longer have the figure of circles, for the motion of oscillation will be propagated through the orifice, and act most powerfully in the direction of a sector, whose centre is at the point whence the original impulse proceeded.

458. When the wind acts to raise waves, they do not diverge from a centre, but usually proceed in parallel lines, straight, or nearly so. If the impulse were momentary, the waves would decrease in height, in consequence of the viscidity of the fluid, and the friction among its particles. But as winds blow for a space of time of some duration, the original impulse is increased rather than diminished, and thus waves continue to rise, and their propagation may take place with increased, rather than diminished altitude. The increase in height will continue, until the sum of the columns elevated above the general level, and the friction become equal to the disturbing cause. The limit to which a single series of waves can be raised by the wind, has been inferred to be no more than 6 feet. As wind blowing over the surface of smooth water, moves parallel to it, the original cause of waves being raised by the wind, is friction; but after the waves are raised, the wind acts upon that surface which is inclined to it; and its force may be resolved into two components, one of which tends to increase the elevation. The whole force of the wind also tends to give a progressive motion, to the mass of water included in the elevation of the wave; and thus the shape of the waves ceases to be a figure, with two surfaces equally inclined to the horizon; and the surface on the side opposite to the wind, has its inclination increased. This increase may become so great as to make the wave project beyond its base; in which case, the force of gravity will cause the summit to break, and roll over the surface of the wave beneath.

If a wind, after having raised a series of waves, shall cease to blow, and another arise from the opposite point of the compass, the latter will act against surfaces more inclined to the horizon, than the other did, and will thus produce a greater effect. It there-

fore happens, that at sea, the highest waves are raised by sudden changes of wind, or when a wind blows in a direction contrary to that in which the motion of oscillation is propagated. Any change in the direction of the wind, will create a new series of waves crossing the first; and thus the elevations and depressions, or the total height of the waves may be increased. It is in this manner that the very great excess of the height of waves beyond the limit stated for a wind blowing in a constant direction, is caused. And in conformity, we find the ocean comparatively smooth in those regions of the earth that are the seat of constant winds, and that the height of waves is greatest in those regions where changes of wind are most frequent.

Wind acting by its friction to raise waves, it may be inferred that if any substance capable of lessening friction be interposed, the elevation that would otherwise take place is lessened. And in consequence, it has been found that when oil is poured upon the surface of water, waves are rarely formed except by the most intense winds; and if poured upon waves already formed, it permits the viscidity and friction of the water to act to bring them to rest; thus, oil may be used to lessen the dangers to which vessels are exposed, by the violence of the oscillation of waves, which is in some cases very great.

459. When waves meet an inclined obstacle, the columns in which we have conceived them to vibrate, are lessened in depth, and thus their fluid pressure is diminished. The waves no longer meeting with the same resistance as before, the liquid acquires a progressive motion, which will carry it up the inclined surface, until its moving force is counteracted by the weight of the quantity thus elevated above its original level. For this reason, breakers or surf, form upon shelving coasts, whatever be the direction of the wind.

When waves are raised by the wind, the influence being exerted wholly upon the surface cannot penetrate to any great depth. From 30 to 40 feet, is inferred to be about the greatest distance from the surface, to which the agitation reaches. It is otherwise with those waves that are formed by the attraction of the sun and moon, and which constitute the tides.

CHAPTER X.

Of the Motion of Elastic Fluids.

460. When an elastic fluid moves through an aperture into a vacuum, it is usually considered as contained in an open vessel, through an orifice in which it passes with a velocity due to a column of the fluid, of sufficient height to give it by its pressure, the density at which it is found. This hypothesis is correct, so far as regards the equality of velocity between an elastic fluid contained in a close, and in an open vessel; provided their densities be identical, for the elastic force is by the law of Mariotte, § 365, exactly equal to the pressure by which any given density is produced.

In consequence of this same law, the density of an elastic fluid, contained in an open vessel, will decrease, as we rise from the surface of the earth; and in order to produce the usual existing pressure, the open vessel must be considered as extending to the utmost limits of the atmosphere. Instead, however, of investigating the circumstances that would actually take place, we consider the elastic fluid as reduced to the liquid state, and as being of uniform density throughout; the height of the column in the vessel therefore becomes that which was, § 357, styled the height of a homogeneous atmosphere.

If we call this height, h, we have from § 407,

$$v = \sqrt{2gh};$$

and taking $h = 27600$ feet, as determined in § 357, we have

$$v = 1328 \text{ feet.} \qquad (508)$$

To take a more exact determination, and which will be applicable to our succeeding researches.

At the temperature of 32°, and under a pressure of 30 inches of mercury, the density of that metal, in terms of air as the unit, is 10467; hence the height of a homogeneous atmosphere, of the temperature of 32°, is 25268 feet, and

$$v = 1295 \text{ feet.} \qquad (508a)$$

461. Air, therefore, of the temperature of 32°, will rush into a vacuum with a velocity of 1295 feet per second. If the temperature be about 60°, the velocity becomes 1328 feet, at which it is usually stated in English books.

It may be at once inferred from this investigation, that when the temperature of air varies, its velocity in entering a vacuum will vary also.

In fact, if m be the expansion of air for each degree of Fahrenheit's thermometer; t, the number of degrees of the thermometer, reckoned from the freezing point, h becomes $h\,(1+mt)$, and

$$v' = \sqrt{[2\,g\,h\,(1+mt)]}\,; \qquad (509)$$

whence, taking the above value of 1295 feet for v, at 32°, we have for v', at the temperature of $t+32°$,

$$v' = 1295\sqrt{(1+mt)}\,. \qquad (510)$$

462. When the density of the air varies, and we abstract the variation of temperature with which such variations are usually attended, the height of a homogeneous atmosphere, of the new density, will be the same as before, and the velocity will not vary.

The velocity of an elastic fluid in entering a vacuum will, by this reasoning, be always the same with that which a liquid of similar density, and capable of exerting an equal pressure with it, would flow from a vessel. And in this form the rule may be extended to the case of air, or other elastic fluids, rushing from a vessel into a space containing an elastic fluid of a different density. The velocity will be in all cases the same as that with which a liquid of similar density, and capable of exerting a pressure equal to the difference of the two tensions, would flow. Thus air, of a tension of two atmospheres having, if its temperature be the same, double the density of that of the atmosphere, will flow out of a vessel into the open air, with half the constant velocity at which air would enter a vacuum.

When air rushes into a vessel in which a vacuum has been previously formed, its velocity is diminished as the vessel fills with air, and should, according to the hypothesis become =0, when the air in the vessel acquires a density equal to that of the air in the space whence it flows. The velocity being considered as due to the altitude of a homogeneous atmosphere, the motion in this case is considered as retarded by a motion growing out of a fall, through an atmosphere of equal and uniform density, by whose pressure the density acquired at the moment in the vessel, would be produced.

In this view of the subject, the velocity with which air rushes into a close vessel, which it finally fills with a mass of density equal to its own, is equably retarded.

463. In gases other than atmospheric air, the velocities with which they enter a vacuum, are in the inverse ratio of the square roots of their densities, for:

If h, be the height of a homogeneous atmosphere of atmospheric air; h', the height of a homogeneous atmosphere of another gas; D, the density of the gas, that of atmospheric air being

unity; the heights, in order to produce the same pressure, must be inversely as their densities, and

$$h' = \frac{h}{D}.$$

Hence, v', the velocity of a gas, whose density is D, will be

$$v' = \sqrt{\frac{2gh}{D}}. \qquad (511)$$

464. This theory is far from being perfectly satisfactory, particularly as it is obvious that the whole of the effects that may be due to the elasticity of the air, are omitted. The most important of these, perhaps, is that which takes place when air rushes into a vessel, in which a vacuum has previously been formed. In this case our theory would appear to show that the velocity is uniformly retarded, until it becomes $=0$; and the density of the air that has entered the vessel the same as that without. This does not occur in practice, for the motion will continue after the densities become equal, and the air in the vessel will be condensed; it will then re-act and expand, and the state of rest will be aquired by a series of oscillations.

465. It has been ascertained by the experiments of D'Aubuisson, that air, in passing through an orifice pierced in a thin plate, is affected like a liquid, and forms a vena contracta, whose area is, as in the case of a liquid, 0.62 of the area of the orifice. The application of cylindric adjutages, increases the quantity that issues to 0.93, and a conical tube to 0.95. The adjutage may be twenty or thirty times the diameter of the orifice in length, before the discharge begins to diminish in consequence of the friction.

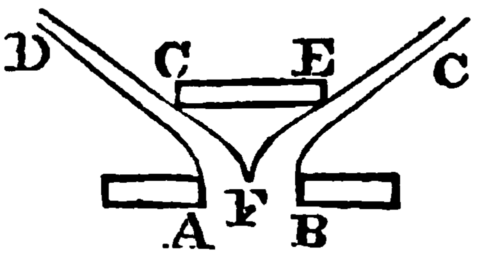
466. The principle, §413, of the lateral communication of motion, holds good in gases, as well as in liquids. Thus, liquids in motion carry with them a current of the air that is in contact with them; and gases, or vapours in motion, carry with them the neighbouring air.

The latter fact may be conclusively established by the phenomena of the Eolipyle. This instrument is a boiler in which steam is generated, and permitted to escape from a narrow aperture. It has for ages been employed to excite combustion. Now steam alone, unmixed with atmospheric air, would extinguish flame, instead of increasing its intensity; and the fact of its being increased, proves that a current of atmospheric air joins the effluent steam, and is carried with it through the burning fuel.

We may upon this principle explain a curious fact, observed in the efflux of air from a bellows, or other machine, in which it is compressed; it has also been observed in the escape of steam from

the safety valves of boilers. If a circular disk of four or five times the diameter of the orifice, be placed close to it, not only will it not be forced away by the current of elastic fluid, but will be retained near the orifice by a force of considerable intensity, in so much, that if the orifice be directed downwards, the disk will be supported in spite of its gravity, even when formed of a dense metallic substance.

That this ought to be the case, may be readily understood when we consider, that an elastic fluid issuing from the orifice, A B, and having its course interrupted by the plate, G E will assume the form of a conoid, D A B C, containing the cavity, G F E; this cavity will at the beginning of the action be filled with a conoid of the elastic fluid.



But if a lateral communication of motion take place, the fluid contained in this conoid will join itself to the stream that escapes at the edges of the plate, and a vacuum will be formed in the conoidal space, G E F; the pressure of the atmosphere acting upon the surface of the plate, G F, will therefore press it towards the orifice. As it approaches the orifice, the action of the fluid will become more intense; for it will strike against the disk; the surface beneath which the vacuum exists, will diminish, and thus the force that acts to repel the disk from the orifice may preponderate, and the disk be forced back; but this force diminishes as the disk recedes, while the surface, to which the atmospheric pressure is due, increases: thus the forces that tend to move the disk in opposite directions, will be continually varying, and under this variation the disk will assume an oscillating motion.

467. Air may not only be set in motion by the difference in pressure, arising from mechanical expansion, or condensation, but also by the physical action of heat, which changes its density.

If by any cause whatsoever, the equilibrium of temperature of a mass of air be disturbed, the parts which are most heated become less dense than those which surround them, and therefore tend to rise; the space that they before occupied, will be supplied by the adjacent air, and thus a circulating motion will take place.

The force with which a portion of a mass of air that has been heated, will tend to rise, is by the principle of § 334, equal to the difference between its own weight, and the weight of an equal mass of the same air, before it was heated.

If the air that is heated be free, it will, both in consequence of the resistance it meets with in rising, and the tendency of elastic fluids to distribute themselves over a given surface, in such manner that the pressure shall become uniform, mix with the air

through which it rises; it will also assume a common temperature with the latter, in consequence of the radiation of its heat.

If the air be heated in a straight tube, or close channel, having an aperture at both ends; and if the two ends are not at the same level, it will rise towards the upper end, and will not mix with other air, or give out much of its heat, until it reach the higher opening. Here it will again tend to mix and distribute itself through the adjacent air. In this manner the motion of air in chimnies takes place.

If air be disseminated through a space unequally heated, and its parts acquire the temperature of the portions of that space which they occupy, a motion of circulation must also take place; and this will continue, so long as the unequal distribution of temperature continues.

In this manner, the currents in the atmosphere, called Winds, are generated, as will hereafter be more fully explained.

468. The density of steam does not vary exactly as its pressure, but follows the specific law stated in § 373. Taking the relative densities and temperatures of steam, as given in the table of § 374, the following results have been obtained.

TABLE

Of the velocity with which steam of different tensions enters a vacuum.

Tension in Atmospheres.	Velocity in feet.	Tension in Atmospheres.	Velocity in feet.
1	1910	5	2040
2	1978	10	2080
3	2007	15	2122
4	2023	20	2142

TABLE

Of the velocity with which steam of different tensions enters a space containing atmospheric air.

T. in At.	V. in Ft.	T. in At.	V. in Ft.
$1\frac{1}{4}$	874	5	1834
$1\frac{1}{2}$	1154	8	1952
2	1400	12	2027
3	1647	16	2070
4	1761	20	2095

CHAPTER XI.

Of the Motion of Gases in Pipes.

469. When the tube in which gases are in motion is long, they are retarded by friction, in a manner analogous to that observed in water and other liquids. Hence, although the velocity of an elastic fluid cannot finally become constant in a tube, many of the circumstances are in other respects similar to those stated in chapter IV. It is however important, that we should investigate them more closely in the case of air.

Let H be the height of a column of mercury that measures the pressure of the air on entering the tube; h, a similar quantity at the place of discharge; b, the altitude of the barometer at the time; t, the height of the thermometer above or below 32°; m, the expansion of air for each degree of the thermometer; d, the diameter of the pipe; V, the velocity; and let

$$T=1+mt.$$

Suppose the mean height of the barometer to be 30 inches, or 2.5 feet.

Taking, as in § 459, the density of mercury, in terms of air, at 32°, to be 10467, the relation of the density of mercury to that of the air that is issuing from the pipe, will be

$$10467\left(\frac{2.5\,T}{b+h}\right). \tag{512}$$

The velocity v, at the place of discharge, will be (509)

$$v=\surd\left(2\,gh.\ 10467\frac{2.5\,T}{b+h}\right); \tag{513}$$

and extracting the square root of the numbers under the radical

$$v=266\ \surd\left(h.\ \frac{T}{b+h}\right). \tag{514}$$

The velocities at different points in the tube will be inversely as the densities of the air at those points; for as the motion is continuous, the same quantity of air must pass through every different section in an equal time. The densities at the ends being proportioned to the pressures, will be proportioned to $b+H$, and $b+h$; the mean density will be proportioned to

$$b+\frac{H+h}{2}. \tag{515}$$

And it will be obvious that the mean velocity may be obtained by an analogy, of which the mean density is the first term; the

pressure at the place of discharge, the second; and the velocity v, the third. Hence, we have for the mean velocity, V,

$$V=266.\sqrt{\left(h\frac{T}{b+h}\right)}\times\frac{b+h}{b+\frac{H+h}{2}} \tag{516}$$

The experiments of D'Aubuisson have shown that the resistance, of which $H-h$ is the measure, is proportioned to the square of the velocity. Other experiments show that it is directly proportioned to the length of the tube L, and inversely to its diameter. Hence, if N be the constant co-efficient of the resistance,

$$H-h=N.\frac{L V^2}{d}; \tag{517}$$

and substituting in this expression the value of v from (514) we obtain

$$H-h=N.\frac{L T h}{d(b+h)}, \tag{518}$$

whence

$$h=H-N.\frac{L T h}{d(b+h)}, \tag{519}$$

or in a more convenient form,

$$h=\frac{H}{\frac{N L T}{d(b+h)}+1}, \tag{520}$$

h being still involved in the second member of this equation, it can only be resolved by an approximation. This may be obtained from knowing that in the cases that most usually occur in practice, $\frac{T}{b+h}$ is a quantity that varies but little, and which may, therefore, without sensible error, be considered as constant. If then we make

$$N.\frac{T}{b+h}=c,$$

we have

$$h=\frac{H}{c.\frac{L}{d}+1}; \tag{521}$$

when h, H and L, are estimated in English feet, and d in inches,

$$c=0.002,$$

and the formula becomes

$$h=\frac{H}{0.002\frac{L}{d}+1}. \tag{522}$$

The pressure under which the air issues, being thus obtained, the effluent velocity is given by the formula (514)

$$v=266\sqrt{\left(h\frac{T}{b+h}\right)};$$

and here again we may make use of a constant quantity for $\frac{T}{b+h}$, without any sensible error.

To find the quantity in cubic feet, the diameter of the pipe being given in feet, the above formula must be multiplied by

$$\frac{\pi d^2}{4};$$

or if the diameter be given in inches, by

$$\frac{\pi d^2}{4\times 144}=\frac{\pi d^2}{576};$$

and calling the quantity discharged Q,

$$Q=1.451\,d^2\sqrt{\left(h\frac{T}{b+h}\right)}. \qquad (523)$$

If we take, instead of $\frac{T}{b+h}$, a constant quantity, determined from experiment in the case of the blowing machines of furnaces, we have, h being estimated in feet, d in inches, and Q in cubic feet,

$$Q=79.44\,d^2\sqrt{h}. \qquad (524)$$

If the area of the aperture, through which the air is discharged, should differ from thc mean area of the tube, as is frequently the case in blowing engines, the velocity may be determined upon the principle, that in different sections of the same tube, the velocities are inversely as the areas, or, if d be the diameter of the orifice of discharge, D that of the rest of the tube, as

$$\frac{d^2}{D^2}.$$

If the orifice of discharge be in a thin plate, this co-efficient would become

$$0.62\frac{d^2}{D^2},$$

and if formed by a cylindrical tube,

$$0.93\frac{d^2}{D^2}.$$

This being the case most usual in practice, we have for the quantity discharged, from (524)

$$Q=74.34\frac{d^4}{D^2}\sqrt{h}. \qquad (525)$$

470. It will be obvious, from what has been stated, that the most important application of this subject is to the tubes, by which air is conveyed from blowing machines to excite the combustion of blast furnaces; in these, a knowledge of the quantity of air they convey, is often of great importance.

Another useful practical case is, that of the conveyance of inflammable gas in pipes, for the purpose of illumination. The above investigations, and the formulæ thence deduced, are applicable in this instance also; for it has been found that the resistance to the motion of carburetted hydrogen in tubes, not only follows the same laws, but is equal in quantity to that which retards the motion of atmospheric air.

The same principles might be applied to determine the velocity of effluence, in terms of the pressure upon the entrance of the tube. The pressure of the issuing air may be easily determined from (522); we have not considered it necessary to enter into the investigation of formulæ for this purpose.

CHAPTER XII.

Of the Motion of Air in Chimnies.

471. A chimney may be considered as a tube, in any position except horizontal, at the lower opening of which air is heated by an extrinsic cause. This air will, in conformity with what has been stated in Chapter X, rise and pass out at the upper opening; its place will be supplied by air pressing from the space adjacent to the lower opening. If this be heated in its turn, as it enters the chimney, to the same temperature, it will rise with a force equal to that possessed by what preceded it; and so soon as the whole tube is filled with the heated air, the velocity will become uniform.

The circumstances will obviously be the same as if a tube were adapted to the bottom of the chimney and filled with air of the original temperature, while the chimney is filled with the heated air; and thus the air will move as in an inverted syphon, in which two columns of fluid, of different densities but of equal altitudes, press against each other. For as the action of the external air may be considered equal on both openings of the chimney, the acceleration it produces in the one column, and the retardation in the other, may be neglected.

Let h be the height of the column of air in the chimney; t the external temperature; t' that of the air in the chimney; m the dilatation of the air for each degree of the thermometer. The length of the column of cold air reduced to 32°, will be

$$h\left(\frac{1}{1+tm}\right). \tag{526}$$

The same column at the temperature t', will be (509),

$$h\left(\frac{1}{1+tm}\right).\ (1+t'm); \tag{527}$$

and this will be the height to which the velocity would be due, with which the air would enter the chimney, if that were void of air; the actual velocity will be due to the difference between this height and h, or to

$$h\left(\frac{1+t'm}{1+tm}-1\right), \tag{528}$$

and we have for the value of v from (509)

$$v=\sqrt{\left[2gh\left(\frac{1+t'm}{1+tm}-1\right)\right]}, \tag{529}$$

or,

$$v = \sqrt{\left[2gh\left(\frac{(t-t')m}{1+tm}\right)\right]}; \qquad (530)$$

the quantity $1+tm$, is, generally speaking, so small that it may be neglected; and we, therefore, have

$$v = \sqrt{[2\,ghm\,(t-t')]}\,, \qquad (531)$$

and we may take for the height to which the velocity is due,

$$hm(t-t').$$

472. Such would be the theory were the air to meet with no resistance in its passage through the chimney. But it is obvious that it will meet with friction; that its temperature and consequent ascensive force will diminish. Thus the velocity with which the heated air would otherwise begin to ascend will be lessened; and its excess of elastic force will continually decrease, from the origin to the summit. At the extremity of the chimney, the elastic force will obviously be proportioned to the excess of temperature that it retains at that point. And as we may consider that the same quantity of air passes through every different section of the chimney in the same time, it follows that at every point the velocity is inversely as the density, and therefore that the velocity decreases from the origin to the summit.

If P be the pressure under which the air enters the chimney; p the pressure it retains at the summit. The loss of motion growing out of resistances in the chimney, may be represented by $P-p$.

The resistance appears from experiment to be directly proportioned to the square of the velocity, and the length of the chimney; and inversely to its diameter.

If p be estimated as the height of a column of the heated air, under the pressure of the atmosphere,

$$p=hm\,(t-t'); \qquad (532)$$

and from (509) and (531),

$$v=\sqrt{2gp}, \qquad (533)$$

calling the velocity V; the diameter of the tube D; its length L; and the co-efficient of the resistance K; the law of the resistance just stated, gives us

$$P-p=K.\,\frac{V^2\,L}{D};$$

whence

$$p=P-\frac{K\,V^2\,L}{D};$$

substituting this value in the foregoing equation, (533), we have

$$V^2=2g\left(P-\frac{K\,V^2\,L}{D}\right); \qquad (534)$$

whence we obtain for the value of V,

$$V=\sqrt{2g}\sqrt{\left(\frac{PD}{D+2gLK}\right)}, \tag{535}$$

and for that of K,

$$K=\frac{2gPD-V^2D}{2gV^2L}. \tag{536}$$

By means of which last formula, the co-efficient of the friction can be obtained by experiment. This has been done by Peclet, who has found when the values of P, D, L, and V, are given in French metres:

(1.) That in brick chimnies,

$$K=0.0127. \tag{537}$$

(2.) In wrought iron flues,

$$K=0.0050. \tag{538}$$

(3.) In chimnies or flues of cast iron,

$$K=0.0025. \tag{539}$$

473. The great difference between the values of the co-efficient of the friction, in tubes of different substances, is a remarkable fact; particularly as it differs essentially from what is observed in the motion of water in pipes, where the substance of which they are composed has no essential influence on the velocity.

This discrepancy may be explained by the difference in the attraction of the two fluids for the substances of which the tubes are composed. Water adheres, by its attraction of cohesion, to the pipes, and moistens them. It thus in fact, in running in tubes, rubs against the water that adheres to them, and the friction is a constant quantity; for it always takes place between surfaces, both of which are composed of the same substance. While in the case of the motion of air, the friction takes place between it and the material of which the tube is composed, and its co-efficient should therefore be different in tubes of different materials.

The dimensions of chimnies depend upon the quantity of air which the combustible requires for its perfect ignition, and upon the velocity which it will assume in them. The former is an object of chemical and physical investigation, and would be out of place in a work on Mechanics. The latter is determined by the foregoing equations.

CHAPTER VIII.

Of the Winds.

474. Winds are currents in the atmosphere, that are generally if not always caused by a disturbance in its equilibrium, owing to the unequal and variable distribution of temperature upon the surface.

The temperature of the surface of the earth, is due to two antagonist causes:

(1). The constitution of the mass of the earth and its atmosphere:

(2). The reception of the radiant heat of the sun.

The heat arising from the first of these causes radiates, and in consequence of this radiation, the earth would continually grow cooler. This waste is supplied by the heat that radiates from the sun. It has been proved incontestibly by Laplace, that the mean temperature of the earth is now constant, and has been so for 2000 years; hence, the quantity of heat that radiates from the earth, and the quantity received from the sun, exactly balance each other; that is to say, for any series of years, the sum of the quantities received by the whole earth from the sun, is just equal to the quantity that radiates.

This equality does not exist for short periods of time. The radiation from the surface goes on continually, although not uniformly, being greatest from the portions that are most heated; while the reception of heat at any given place, only takes place while the sun is above the horizon.

The quantity of heat received from the sun, upon a given surface, varies in a given natural day with the altitude of the sun above the horizon, in consequence of the greater or less obliquity of his rays. It also differs from day to day, in consequence of the variation in the length of the natural day, and in the meridian altitude of the sun.

The first of these variations grows out of the rotation of the earth upon its axes, in the space of a day; the second, arises from the revolution of the earth around the sun, in an annual orbit inclined to the equator.

This orbit being an ellipse, of which the sun occupies one of the foci, and the arcs of the ellipse described not being proportioned to the times, the sun's daily apparent motion is not equable.

The orbit being inclined to the axis of rotation, causes a variation in the length of the natural day, and in the sun's meridian altitude; hence flow the vicissitudes of the seasons. The diameter of the orbit that passes through the sun, does not pass through the equinoctial points, but is nearer to the solsticial points than to the equinoctial; hence the equinoxes do not divide the year into two equal parts, but the summer of the northern hemisphere is about $7\frac{2}{3}$ days longer than the summer of the southern. It follows that the northern hemisphere receives more heat from the sun, than the southern; and a balance taking place in the quantity of heat received and radiated, not only in the whole earth, but in the two separate hemispheres, the northern hemisphere is warmer than the southern; therefore, even the mean place of the equator of temperature does not coincide with the astronomic equator, but lies north of it.

Did the plane of the earth's orbit coincide with the terrestrial equator, the days and nights would be constantly equal in every part of the globe. The sun's rays at noon would fall vertically upon points in the equator, and would be tangents to the earth at the poles; and were there no lateral communication by radiation or the conducting property of the materials of the earth, and its atmosphere, the quantity of heat received at noon, from the sun, at every different place, would obviously be proportioned to the cosine of the latitude. The mean temperature of the days in every latitude would be constant; and this mean diurnal temperature would follow a regular law of decrease, from the equator to the poles. Taken for a whole year, the mean temperature, if disturbing causes did not act, should follow a similar law. The variations at the surface are so sudden as to cloak this law, except when studied by the comparison of a long series of thermometric observations; but when we penetrate beneath the surface of the earth, to a depth sufficient to be removed from the influence of the changes at the surface, we find this law to hold good:

The temperature at the depth of from 60, to 80 feet beneath the surface of the earth, is constant in every different climate, and corresponds closely with the mean temperature at the surface.

At the equator, the days and nights are of equal lengths throughout the year; and the meridian zenith distance of the sun, never amounts to more than $23\frac{1}{2}°$. At the polar circle, the natural days vary in length from 24 hrs. to 0 hrs., and the change in the meridian altitude is from 0° to 47°. Thus, at the equator, the temperature varies but little on each side of the mean; while, were the earth of uniform surface, the extent of the variation on each side of the mean rate should increase regularly from the equator, to the polar circles. Within the polar circles, the sun does not

rise for several days on each side of the winter solstice, and is above the horizon for several days near the summer solstice; while at the poles, the natural day and night have each a length of six months. Hence it might be inferred, that even greater variations on each side of the mean temperature should occur within the polar circles, and that the variation should be a maximum at the pole.

The quantity of heat derived from the sun on the day of the summer solstice, has been calculated to be nearly equal in the lats. of 45° and 60°; and when the sun's declination exceeds 18°, the quantity of heat received in 24 hours at the pole is not less than it is at the equator, during the twelve hours the sun is above the horizon.

It thus happens that were there no disturbing cause, nor any means by which the excess of heat might be conveyed from one region to another, the distribution of heat at the surface would be continually varying. The distribution according to a regular law of decrease, from the equator to the poles, would only take place near the time of the equinoxes; while at the solstices the parallels receiving the greatest quantity of heat would be without the tropics; and parallels in the frigid and torrid zones might receive equal quantities of heat on the same day.

As however the earth has at no great depth, the mean temperature of the climate, this will tend in a high latitude to prevent the surface from acquiring a heat as great as that actually communicated on the hottest days; and thus the heat of the surface in such latitudes will never rise as high on the warmest days, as is consistent with the quantity actually received. In the same manner the surface of high latitudes never cools as low as is consistent with the quantity of radiation from the surface, in the absence of any supply from the sun.

Local circumstances that will hereafter be stated, affect the range of sensible heat; and thus, places in the same latitude, many have very different maxima and minima of temperature; and the amount of variation may be much greater in a given place, than it is in another of the same latitude.

Certain physical causes interfere in high latitudes, to prevent the extent of the changes of temperature being as great as they would be, in consequence of the great difference in the altitude of the sun, and of its continuance above the horizon at different seasons; these will be stated in their proper place; and thus the greatest alternations seem to take place, in the lat. of from 35° to 50°. In New-York, the annual range of the thermometer, from its summer maximum to its winter minimum, sometimes exceeds [illegible]°; and the difference between the mean temperature of the

hottest, and that of the coldest month, amounts to 56° ; at Pekin, the latter difference is 60°, while at Funchal, it is no more than 10°.

475. Difference of elevation above the surface of the earth, has a great effect upon the temperature of places. The air of the atmosphere is from its elastic nature, denser at the mean surface of the earth, than in higher regions ; and air has an increased capacity for heat when it becomes rarer ; hence, in the higher parts of the atmosphere an intense cold prevails, and the temperature of the land decreases with its elevation above the level of the ocean. So intense is the action of this cause, and so speedily is it sensible in rising from the earth, that even in the heart of the torrid zone mountains exist whose tops are covered with perpetual snow. On these, it is therefore evident, that the thermometer never rises much above 32°.

It has been inferred, but without sufficient reason, that the mean temperature of the limit of perpetual snow is 32°, but observation shows that it is in all cases lower, and the limit appears to arise rather from the mean temperature of the warmest month, than from that of the entire year.

It has been estimated that the temperature decreases as we recede from the surface of the earth, at the rate of about 1° for every 270 feet.

In consequence of the great variation that takes place in the quantity of heat received from the sun, in temperate and frigid climates, while the radiation has a much less range, an accumulation takes place, at those seasons when the sun is highest at noon, and remains longest above the horizon, by which the temperature increases for some time after the solstice ; a corresponding diminution in temperature goes on after the shortest day. Thus it happens that the greatest heat in middle latitudes occurs about a month after the summer solstice, and the greatest cold about an equal time after the winter solstice.

An empirical formula, that very nearly corresponds with observation, has been framed to represent the temperatures at different seasons, and at altitudes above the level of the sea, in all latitudes.

Let M be the mean temperature in lat. 45° ;
M+E, the mean temperature at the equator ;
L, the latitude of the place ;
F, a co-efficient determined by observation ;
H, the altitude of the place above the level of the sea ;
l, the sun's longitude.

Then we have for the mean diurnal temperature, on the day for which the longitude l is given,

$$t = \mathrm{M} + \mathrm{E} \cos.\, 2\,\mathrm{L} + \mathrm{F} \sin.\, (l - 30°) - \frac{\mathrm{H}}{270}. \qquad (540)$$

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If $F = 15^{\circ}$, the formula gives results that are on the average true, in the western part of Europe, and in the North Atlantic.

476. The nature of the surface has a great effect upon the distribution of temperature, and upon the distance that exists between the extremes of heat and cold in different parts of the globe.

The surface of the earth is partly of solid land, and partly water. Within the former, the communication of heat is extremely slow, and hence the surface of the land adapts its temperature more closely to the quantity of heat received daily, than the surface of the ocean. The latter, when exposed to heat that varies from place to place, is set in motion; for so long as the temperature of the surface does not fall below 40°, the water expands, and the columns in the warmer parts increasing in altitude, while they diminish in density, a current is caused from the parts most heated, to those which are colder; a counter current is formed in the water beneath, in which the colder portions flow towards the zone of greatest heat. In this manner, so much of the heat derived from the sun, as exceeds the radiation, is conveyed at the surface, from the heated regions, to those which are colder.

This motion ceases, however, when the temperature of the surface falls below 40°, beneath which degree any diminution of the temperature of the water will render it lighter than that which is beneath, and the heated portion sinks, instead of rising.

When the sun shines upon the land, its calorific rays penetrate to but a small depth, say no more than a few inches; its surface is in consequence rapidly heated, when the heat received exceeds that which is radiated: when the latter is in excess, the loss of heat is principally confined to the surface, which is therefore rapidly cooled.

In water, when the reception of heat exceeds the radiation, the calorific rays penetrate to a considerable depth, say 20 to 30 feet; the heat being thus distributed through a large mass, the superficial temperature is but slowly altered. When on the other hand, the radiation is in excess, the upper portions, on parting with their heat, contract, and becoming heavier than the water which is beneath, descend until they reach the bottom, or a stratum of the fluid of equal temperature with themselves; a circulation is thus kept up, and the heat lost, although equal, or even superior in quantity to that withdrawn from the land, is again derived from a large mass; the diminution of the superficial temperature is therefore slow. When however the surface is cooled below 40°, this motion ceases.

From the combination of these circumstances, it happens that the temperature of the surface of the ocean is more constant than that of the land; that it can be reduced to certain laws easily dis-

covered from observation; and that it follows much more closely than the land, the law of a regular diminution of temperature, from the equator to the poles. The variations on each side of the mean temperature, are also less on the ocean than they are upon the land. These rules likewise hold good in islands, and to a less extent in countries adjacent to the ocean; these portions of the land have climates of less vicissitude than the interior of continents.

477. Great and deep lakes have a similar, although less important influence on climate; for although the extent of their surface be not sufficiently great to cause any distribution of heat by currents, the difference between the quantities of heat received and radiated, affect not their surface alone, but their whole mass. Their surface, therefore, like that of the ocean, preserves a more uniform temperature than that of the land.

When a lake cools, the motion that we have described in speaking of the ocean, in which the cooler parts descend, and by which the heat is withdrawn from the whole mass, goes on until the temperature throughout becomes 40°. Water at this temperature reaches its maximum of density, the motion of descent ceases, and the surface will be speedily cooled to the temperature of congelation. Deep lakes, however, descend to such depths as to come into contact with those strata of the earth's mass that retain the mean temperature of the climate; from these the water will derive heat; and thus it may happen that a deep lake, of no great superficial extent, is never frozen. Such phenomena occur in the small lakes of the western part of the state of New-York, the surface of which never freezes.

Shallow lakes and morasses tend to make a climate colder; for the cold produced at their surfaces not only by evaporation, but by radiation, cannot long be compensated by an internal motion.

The draining of morasses, renders a climate warmer, as does the cutting of forests, and the extension of cultivation. The effects of the latter causes appear to extend beyond the region where they operate directly. Thus, the cultivation of France and Germany, has changed the climate of Italy; and thus, the clearing of the forests of the interior of the United States, has raised the mean annual temperature of the seacoast.

478. To recapitulate our general inferences:

(1). Upon the land, the zone of greatest sensible heat will be a little north of the equator on the days of the equinox; but will on other days of the year vary in position; and will be found in the interior of continents about a month after the solstices, in latitudes as high as from 40° to 50°. In the ocean, on the

contrary, the zone of maximum temperature does not vary in its position more than 8°, and is always to the north of the equator. From this zone, the heat of the surface of the ocean decreases uniformly to a latitude of from 28° to 30°. Beyond this limit, on either side of the equator to the latitude of 50°, the heat of the surface is alternately greater or less than would be consistent with a regular decrease, according to the law of the cosine of the latitude; but after the equinoxes, it appears to coincide for a short time with the results of that law.

(2). Elevated countries are colder than those more near the level of the sea.

To these, it is to be added, that the western sides of the two great continents are sensibly warmer, or have a higher mean temperature than the eastern.

(3). The change in the density, caused by change of temperature, produces currents in the ocean; the surface of water also becomes more slowly heated, and parts with its heat less rapidly than the surface of land exposed to an equal action of the sun's rays. The ocean therefore enjoys a more equable temperature than the land, and influences in a similar manner the climate of islands and seacoasts.

(4). Cultivation appears to raise the mean temperature, and certainly ameliorates the climate. In the United States, this effect appears to be well marked, but is attended with an anomaly. The duration of intense cold has been sensibly lessened, but the diminution of the length of the winter is wholly in its earlier part; on the other hand, frosts are experienced at later dates in the spring than formerly. Such are the more important circumstances that influence climate, and on them a theory of the winds may be founded.

479. The air of our atmosphere receives heat from, and communicates it to, the parts of the earth on which it presses. Those parts of it in immediate contact, acquiring or parting with heat readily; their volumes and tensions are therefore changed, a disturbance of equilibrium takes place, and motion ensues. Thus fresh portions of air are brought into contact with the surface of the earth, and the influence of its changes of temperature extended. These motions in the atmosphere, concur therefore with those of the ocean, of which we have already spoken, to moderate the vicissitudes of heat, to which the surface would otherwise be subjected.

The lower stratum of the air of the atmosphere, tends in consequence, to an equilibrium of temperature with the surface beneath it; this state it however never reaches, or never retains for more than a short space of time; besides, in its own tendency

to move, until a state of equilibrium of temperature be attained, it is set into a continual, and frequently violent motion.

This state of equilibrium, it may be stated, is not that of uniform temperature throughout; but would be one of uniform temperature at the mean surface of the earth, and of a temperature regularly decreasing from that surface upwards, in conformity with the relations of the air's diminishing density to specific heat.

Had the air no motion growing out of such disturbances of temperature, its inertia, and the friction that takes place between it and the earth, and among its own particles, would cause it to assume precisely the same angular velocity with the part of the surface immediately beneath it. In its motions it must therefore be considered as acted upon by two forces; the one arising from the disturbance of the equilibrium of temperature; the other, from the rotary motion of the parallel whence it begins to move over the surface.

From the foregoing considerations it will be seen that the earth's atmosphere must be in a state of almost constant motion, forming the currents that are styled Winds.

Upon the greater part of the surface of the ocean, these are reducible to fixed and determinate laws. Upon continents, and in high latitudes upon the ocean, although we may assign the general causes of the winds, yet the order and periods of their recurrences are irregular.

480. The winds may be divided into classes, which we shall enumerate before proceeding to explain their causes. They are

1. The Trade Winds;
2. Monsoons;
3. The local variations of the Trades and Monsoons;
4. The regular Westerly Winds;
5. The variable winds of continents, and of temperate and polar climates;
6. The land and sea breezes.

The theory of the winds has derived most important accessions from the researches of Daniell, whose labours we shall make use of in the explanation of these phenomena. As it is unnecessary to enter into any strict calculations in relation to them, we shall, in this discussion, dispense with the use of algebraic notation.

481. Were the earth a sphere of uniform temperature, and at rest in space; its atmosphere a perfectly dry and permanently elastic fluid; the height of the latter would be constant over every point of the earth's surface, and its density and elasticity, at equal elevations, every where the same. The column of mercury that it would support in the barometer, would therefore be the same at

every point on the surface of the sphere; and equal at equal heights above the surface. The atmosphere would be absolutely at rest; and as its elasticity is proportioned to the pressure, the density would decrease in geometrical progression, while the distance from the surface of the sphere increases in arithmetical. When air is rarefied, its capacity for heat is increased, and *vice versa;* the sensible heat of the atmosphere must therefore decrease as the altitude increases; and as this changes the volume of elastic fluids, even under equal pressures, the barometer alone will no longer be the exact measure of the progressive density, but must be associated with the thermometer. Any change of temperature that affects every part of the sphere, would cause an increase in the elasticity of the atmosphere, and in its consequent height, without producing any motion in the lateral direction, or any change in the pressure upon the surface; but the pressure will be changed at all other altitudes.

If the temperature of the sphere, instead of being equal at every point, were greatest at the equator, and decreased towards the poles, the pressure on every point of the surface would still continue the same; but the altitude of the atmospheric column would become greatest at the equator, and its specific gravity at the surface less there than at the poles. The heavier fluid at the poles must, by its greater weight pass beneath, and displace the lighter, and a current will be established in the lower part of the atmosphere, from the poles towards the equator. The difference in the specific gravity of the polar and equatorial columns becomes less as we ascend into the atmosphere; while the elasticity, which is constant at the surface, varies with the height, and the barometer stands higher at equal elevations in the equatorial, than in the polar column. It will hence happen, that, at some definite height, the unequal density of the lower strata will be compensated; and a counter-current will take place in the higher regions from the equator towards the poles.

The heights at which this would happen, under certain circumstances may be calculated, and the velocity of each current determined. This has been done by Daniell, to whose work the reader is referred, for the process and inferences. From his investigations it appears, that the lower current directed from the poles towards the equator, extends to the height of two miles and a half, gradually diminishing in velocity from the surface upwards. At the last mentioned height the counter current begins, and its velocities gradually increase from that altitude upwards.

The velocity and direction of these currents may be affected by the partial rarefaction or condensation of any of the columns; and such change of density will naturally take place, in consequence

of the vicissitudes of the seasons, and the alternations of day and night.

If the sphere revolve around its polar diameter, as an axis, an apparent modification will take place in the direction of the currents. The lower current, coming from a point whose velocity of rotation is less than that at which it arrives, will appear to be affected with a motion, in a direction contrary to that of the revolution of the sphere; while the upper current, being under opposite circumstances, will be apparently affected in an opposite manner.

The earth revolves around its axis in a direction from west to east; and hence the great equatorial currents, that are in fact directed, on the north side of the equator, from north to south, and on the south side from south to north, appear in both cases deflected towards the east.

In the months of April and October, such a state of things does actually take place upon the earth; hence N. E. winds prevail at those periods throughout the whole northern, and S. E. winds throughout the whole southern hemisphere; the hemispheres being divided by the equator of temperature, and not by that of latitude.

At other seasons, the regular law of decreasing temperature is interrupted even upon the surface of the ocean, at latitudes of from 28° to 32°; and is not to be recognised upon the land; hence these winds are constant only within these limits, and in the open ocean. These constant currents are called the Trade Winds, and from their directions, the N. E. and S. E. Trades.

The velocity of rotation changes more for a given difference of latitude in high than in low latitudes; hence the apparent deviation, from a true northern or southern direction, will be greatest near the outer verge of the trade winds, and least near their central zone.

482. This central zone that divides the trade winds, has a breadth, varying at different seasons, from 2½ to 9 degrees. It corresponds with the equator of temperature, and hence varies in position, § 474, a few degrees, but is always on the northern side of the equator. Within this narrow zone the winds are subject to no regular law, and hence it is said to be the seat of the Variables. In this space the velocities of the currents proceeding in opposite directions destroy each other, and an accumulation, as has been stated, would take place, did not the air rise and join the counter-current, that continually flows in the higher regions.

At the outer limits of the regular trades, it might be inferred that the descent of the counter-current would form a narrow zone,

of winds uncertain in direction, and generally light; such a zone is distinctly marked, and well known in the North Atlantic Ocean. Whether it be found in the Pacific and South Atlantic Oceans cannot be stated, for the want of careful and sufficient recorded observations.

The trade winds, as may be inferred from this theory, prevail in the open ocean, in the Atlantic and Pacific, between the latitudes of 30° N. and 27° S. On entering them from either side, the deviation, growing out of the rotation of the earth, towards the east, is greatest; and this deviation becomes less and less as the equatorial zone is approached; in the immediate vicinity of this zone, the wind is nearly due N. on the northern side of the equator, and nearly due S. on the southern.

483. In the Indian Ocean, winds changing their direction half yearly, and blowing regularly in each direction for nearly six months, are experienced. These periodic winds are called the Monsoons. The cause of them is to be found in the position of this ocean in respect to the adjacent continents.

To the north of the Indian Ocean extends the whole mass of the old continent, with the exception of the southern extremity of Africa. The ocean and the land, thus placed, are acted upon by the sun, in his annual course, with different degrees of intensity at different seasons. In the summer of northern latitudes, the sun is vertical over large portions of the continent, and according to the principles of § 476, the superficial heat of the land being more speedily raised, even by an equal exposure to the sun, becomes greater than that of the ocean. The denser air at the surface of the ocean therefore presses towards the land, causing a current whose absolute motion is from south to north. On the northern side of the equator, coming from a point whose velocity of rotation is greater than that of the points it meets in its course, it has an apparent deflection towards the west and forms a S. W. wind. Hence in the Indian Ocean, the south-western monsoon blows between the months of April and November, on the north side of the equator.

The causes that produce the S. W. monsoon, also operate on the southern side of the equator, as far as 11° S. The current they cause, pressing N. towards the Equator, from a parallel that has a less velocity of rotation, appears as a S. E. wind.

When the sun is on the southern side of the equator, the old continent, losing by radiation more heat than it receives, becomes colder than the Indian Ocean. The air above it, therefore, presses to the south, and the influence extends as far as 11° S. For reasons the converse of the preceding, the apparent direction becomes N. E. on the northern side of the equator, and N. W.

on the southern. To the south of this parallel the regular S. E. trade wind blows continually, in the Indian as well as in the other Oceans.

484. The most important modifications of the trade winds, growing out of local circumstances, are as follows:

The continent of Africa, over which the sun is continually vertical, is always heated at the surface, for reasons already assigned, § 476, to a temperature higher than the adjacent ocean. Hence, in the Gulf of Guinea, a wind sets almost constantly towards the land, and is modified in its direction by the tending of the shore. Between the region in which this sea breeze blows, and that in which the trade winds begin again to prevail, these two winds, diverging from the same space, cause an exhaustion, which is supplied by a counter-current in higher parts of the atmosphere. Within this interval there is a portion of the surface of the ocean that is the seat of almost perpetual calms.

The course of the trade winds is interrupted by the continent of America, hence their influence is not felt until at some distance from the coast of the Pacific Ocean.

Upon the eastern coast of North America, in the summer season of the respective hemispheres, the greater heat of the land draws a current from the ocean; by this the extent of the trade wind is increased. The course of this part is, however, E. or even S. E. on the coast of Florida, Georgia, &c.

The monsoons in the neighbourhood of the land, have their courses deflected also, and sometimes their influence merges altogether in the land and sea breeze.

485. Between the parallels of 30° and 40°, in both the North and South Pacifics, a westerly wind blows almost constantly; intermitting only for a short space of time, after each equinox, when a regular distribution of temperature, over the whole earth, gives rise to the N. E. and S. E. trade winds.

In the Northern Atlantic this wind is not constant, in consequence of the comparative want of breadth of that ocean, by which it is subjected to the influence of the contiguous continents. A westerly wind is, however, the prevailing wind in this ocean, except in the months of April and October, when a N. E. wind is more frequent.

The cause of the existence of such westerly currents may be thus explained.

Within the tropics, and to a short distance beyond them, the variations of temperature, from a law of regular decrease from the equator towards the poles, are so small as to be insensible, and hence, as has been stated, the trade winds are constant within

certain limits. Without these limits, the parallels are alternately warmer and colder, according to the season, than would be consistent with the law of regular decrease. In some one parallel, the deviation from this law will be the greatest. This may be taken as about the parallel of 40°, in which, as may be seen from the examples of New-York and Pekin, the vicissitudes of temperature are excessive. When this parallel is more heated than is consistent with the law of the mean temperatures, the course of the great current from the poles towards the equator is interrupted, the atmosphere in contact with the surface of the earth will be accelerated on the side of this parallel nearest to the pole; on the side nearest the equator, the air increases in density, and hence moves in a direction contrary to that which it would have if the temperature decreased regularly towards the poles. To counteract the condensation that would hence arise in the parallel of 40°, a counter-current takes place. The lower current coming from a parallel whose velocity of rotation is greater than that which it reaches, is apparently impressed with a motion from W. to E. and becomes a S. W. wind on the north side of the equator, and N. W. on its southern side.

When this parallel has a lower temperature than is consistent with the law of regular decrease, it has been demonstrated by Daniell, that the atmospheric pressure would be diminished; for this reason a current would set toward it on both sides, in order to restore the equilibrium, and thus the two causes so different in themselves, will produce similar effects, and winds deflected towards the west will again take place.

The slow changes that take place in the temperature of the surface of the ocean, growing out of the causes stated in § 476, make the parallel in which these opposite influences operate to produce this effect, nearly constant in position. And it is for a similar reason, that the monsoons do not vary gradually in intensity and direction with the declination of the sun, but intermit wholly for a time, and then assume the new direction.

The interval of the monsoons is attended with great oscillations in the atmosphere; great accumulations take place in some places, attended with corresponding rarefactions in others; these mutually re-act upon each other; thus violent storms, the Typhoons of the Indian seas, occur in the interval of the monsoons.

486. Upon the continents, the changes of temperature from day to day, and the alternations of heat from day to night, are rapid and frequent; hence there is no constancy in the direction or intensity of the winds. In high latitudes, even in the open sea, similar inequalities occur. Hence, the land and ocean, in latitudes higher than 40°, are the seat of winds that can be reduced

to no fixed laws, and the frequency of whose changes increases with the latitude.

In the blowing of these variable winds, the inertia of the air tends to cause accumulations in the parts towards which they blow, and expansions in those whence they come; the elastic nature of the air allows these to increase, until the moving force is destroyed, when a returning current is formed which will again cause similar condensations and exhaustions. Thus the variations in the height of the barometer, which have been noted in § 385, become of greater extent in high, than in low latitudes; and when winds have gradually expended their force, a wind in a direction exactly contrary often succeeds.

Although the variable winds of temperate climates are subject to no fixed laws, still we may often find in the local circumstances of countries, reasons why certain winds should blow more frequently than others. Such winds are called the Prevailing Winds, of the particular climate.

In the northern and middle portions of the seaboard of the United States, the great prevailing winds are the N. W., the N. E, the S. W., and the S. E. By an attentive examination of the circumstances of the country, we may easily show why these should be of frequent occurrence, and probably prevail to the exclusion of all others.

A great current of the ocean called the Gulf Stream, proceeds from the Gulf of Mexico, and runs nearly parallel to the coast of North America, as far as the banks of Newfoundland. This current during the winter months, is much warmer than the neighbouring continent; hence a current of air frequently sets from the land towards the ocean, which forms the N. W. wind of the United States.

In summer, although the land becomes warmer than the Gulf Stream, the great difference of temperature between the seaboard and the interior, in which at the lat. of 60°, and at the depth of six feet beneath the surface, the ground is entirely frozen, is sufficient to account for the N. W. being a frequent wind.

The trade winds are interrupted in their course by the great chain of mountains that traverse nearly the whole continent of America. This interruption causes an accumulation of air against their sides. It cannot be lessened by a return in the direction whence it came, in consequence of the perpetual current of the trade winds. It will, therefore, when no other cause for a prevailing wind exists, press over the whole continent; and, if the accumulation have been going on for a long time, will exert a force that no other wind of our climate does.

The N. E. wind of the United States, as well as that of Europe,

may be considered as the great current, directed towards the equator, and exerting its influence when no other cause is in action. That it should be frequent, may be explained from the fact, that the cleared and cultivated portions of the United States, will often be under the circumstances of a regular increase of temperature, from the N. E. to the S. W. This wind was formerly confined to a strip extending but a few miles from the coast. As the country has been cleared, its influence has been more extended, and it is annually becoming more prevalent.

If a N. W. wind have prevailed for some time, as a current of air must, according to the law of lateral communication of motion, follow the direction of the Gulf Stream, an accumulation must take place over the latter, which, seeking to restore the equilibrium of pressure, presses towards the continent, and causes a S. E. Wind.

487. The land and sea breezes are winds whose period is a single day, the former prevailing when the sun is below, the latter when he is above the horizon. The causes are to be sought in the different manner, in which land and water are affected by radiation, and the direct heat of the sun. During the day, the surface of the land, as will be seen by reference to § 170, becomes more heated than that of the adjacent ocean. Hence a current beginning at some hour in the morning, and continuing until the sun is near setting, will flow from the water towards the land. At night the water remains warm, while the surface of the land cools rapidly, and hence the current sets from the land towards the water.

Of all the winds in the climate of New-York, a north wind is perhaps the most rare. It however sometimes blows for two or three days together, and is remarkable for the extreme heat with which it is attended. The cause of this high temperature seems to be, that a north wind prevents the action of the sea breeze, that would otherwise act to temper the climate, and moderate its vicissitudes.

The variable winds of temperate climates, as may be seen from what has been said, arises either from a condensation in some part of the atmosphere, or a rarefaction in others, or both may concur.

When the former is the cause, the wind proceeds forward, extending itself in the direction towards which it appears to blow. Such is the case with the N. W. wind of our climate.

Within the tropics, the islands and countries situated upon the sea-coast, have their climates tempered by winds styled the Land and Sea Breezes. The sea breeze begins to blow three or four hours after sunrise, and continues to blow until a little after sunset. The land breeze commences three or four hours after

sunset, and continues until about sunrise. The cause of these alternating winds may be thus explained: The land and sea, being both equally exposed to the action of the sun during the day, the former as explained in § 476, becomes more heated at the surface than the latter. The air in contact with the land, in consequence expands and rises, while that over the sea presses in to supply the place. At night, the surface of the land parts with its heat most rapidly, and the course of the current is reversed.

These winds are not confined to the tropics, but may be observed on the sea-coast of countries situated in latitudes as high as 45°, during the summer of the hemisphere in which they are situated. Thus, at New-York, a sea breeze is experienced almost daily during the months of June, July, and August; and a land breeze may be occasionally observed during the same months.

When the latter cause occurs, the wind will appear to blow first in the quarter towards which it is directed. Such is the N. E. wind of our climate, which begins to blow in Florida many hours before it is felt at Boston.

The manner in which a wind of this character may arise, and thus extend itself *to windward*, may be illustrated by referring to what occurs on opening the gate of a sluice, in which case the current sets towards the opening; but the motion begins in immediate contact with the sluice, and is propagated in a direction contrary to the current.

CHAPTER XIV.

Of the Motion of Vapour in the Atmosphere.

488. It will be seen by reference to § 370, that water forms vapour at all temperatures whatsoever, of a tension and density having relation to the temperature, according to the tables of § 374 and § 379. Now, as a great portion of the surface of the earth is covered with that liquid, it follows, that vapour will rise from it, and by the general property of elastic fluids to form atmospheres independent of each other, will tend to distribute itself over the surface of the earth; and it would assume the tension and elasticity due to the temperature of the space it occupies, did no opposing force act to prevent it.

489. Were the earth a sphere, wholly covered with water, of uniform temperature throughout its surface, and if we suppose the aerial atmosphere not to exist, the water would form an atmosphere of vapour, whose pressure would be equal to the elastic force of vapour at the constant temperature of the surface. The temperature of this aqueous atmosphere would not be uniform throughout, but would be so only at the surface, for the higher portions undergoing less pressure would expand, and their temperature would diminish to that corresponding to the tension and density of the vapour at the given point. To take an instance: Were a sphere whose surface is wholly covered with water, and which has no aerial atmosphere, to have at every point on its surface the temperature of 32°; an atmosphere of vapour would be formed around it, whose tension would be 0.2 inches, and whose temperature at the surface would be 32°. At the altitude of 30,000 feet it may be calculated that the tension would be diminished one-half, or to 0.1 inch, and the temperature of the vapour to 13°.

The atmosphere of vapour would be in perfect equilibrium, and at rest, over the whole surface of the sphere; and would, by its pressure, prevent the formation of any more vapour. No precipitation would occur in any part, and the whole mass would be clear and transparent.

An uniform increase of the temperature of the surface, would cause the formation of new vapour; the tension of the whole would become uniform; the temperature of its lower parts would be the same as that of the surface of the sphere, and an analogous decrease of temperature and tension would take place at increasing elevations.

490. But if the temperature of the sphere were to become unequal, the circumstances would be different. If we assume it to follow the law of its mean temperature, being warmest at the equator, and to decrease in heat according to some definite law, from the equator to the poles; the tension of the vapour over the whole surface, would be due to the minimum temperature, § 372, or to that of the poles; but the evaporation being due to the heat of the different points on the surface, would be determined by that heat, and go on continually; while at the poles, an equal and rapid condensation would take place. The vapour would in this case, flow in mass, from the equator towards the poles, and the precipitation at the latter points would raise the level of the ocean, until currents were formed, by which all the condensed water would flow back to the equator.

491. If some retarding force were to act, by which the flow of vapour is resisted in its course from the equator towards the poles; the precipitation would be distributed throughout the whole sphere, except at the zone of greatest temperature. Continual evaporation would go on at the equator; continual precipitation at the poles, and both evaporation and precipitation at all intermediate latitudes.

Such a retarding force is to be found in the aerial atmosphere. The vapour, although it constantly tends to form an atmosphere, according to its own mechanical laws, is resisted in its motions by the aerial atmosphere through which it is compelled as it were to filter; and thus, were the circumstances of which we have already spoken, to be all that affect the mixed mass of air and vapour, condensation would be taking place in the higher regions at all latitudes, attended at the same time with evaporation from beneath.

492. The relations of air under different pressures to heat, are different from that of vapour, and the temperature of an aerial atmosphere diminishes much more rapidly with increasing elevations, than that of an aqueous atmosphere; here then, we find upon the principles of § 368, a new cause of precipitation, which would in the higher regions be attended with a corresponding increase in the evaporation from beneath.

If then the earth were wholly covered with water, continual rains, or at least perpetual clouds, would be experienced every where but at the equator.

493. As, however, rather more than one-fourth part of the earth's surface is dry land; this produces a very marked change in the circumstances of the atmosphere. The land furnishes a comparatively small quantity of vapour. Hence, as vapour tends

to form an atmosphere of itself, distributed according to the relations of temperature and tension, over the whole surface, the vapour formed over the surface of the ocean would continually press towards the land, until a state of equilibrium could be attained.

If the land be warmer than the ocean, the vapour would be heated above its original temperature, and a greater quantity by weight, could exist without deposition in a given space; hence, the vapour that might otherwise be precipitated over the ocean, would be diverted towards the land, and even there no deposit might ensue.

If the land be colder than the adjacent ocean, vapour will still flow towards it, but it will now be condensed upon it, and a part at least of the condensation that would otherwise take place upon the ocean, will take place upon the land.

The ocean has a temperature far less variable than that of the land, and thus both will be affected with alternations of rain and sunshine, according to the relations between their temperatures. In the temperate and frigid zones, these will be subject to no fixed laws; but in the torrid zone, seasons of considerable length will be wet or dry, according to the latitude of the place and the declination of the sun.

494. The flow of the vapour, in conformity with its own mechanical laws, is not only retarded by the mere resistance of the atmosphere, but is affected by the winds. When the course of the wind coincides with the direction the vapour would assume under its own pressure, the flow of the vapour is accelerated; when the contrary is the case, it is retarded; and it may thus happen, that some districts of the continents are wholly deprived of moisture, while others receive a superabundant proportion.

Winds also agitate and mix together masses of air of different temperatures, and when these contain moisture, of a tension approaching to the maximum due to their respective temperatures, precipitation must almost always ensue. That this must be the case, will appear from the consideration, that the quantity of vapour that can exist in a given space, varies in geometric progression, while the temperature varies in arithmetic; and the temperature that results from the mixture of equal masses of air, is the arithmetic mean of their respective temperatures. As the latter is always greater than the geometric mean, the quantity of vapour, if both masses of air approached to saturation, will be greater than is consistent with the resulting temperature, and the excess must be precipitated.

495. It will be obvious, from what has been stated, that vapour is in almost all cases pressing from the ocean towards the land;

while upon the latter a precipitation must ensue, often greater than upon an equal surface of the ocean. In this excess of precipitation we are to seek the origin of springs and rivers; by the latter this excess is restored to the sea, to be again evaporated, and thus keep up the continual circulation.

496. So long as aqueous matter remains in the state of vapour, it is transparent. On its first condensation a cloud appears. The manner of the formation of clouds is as follows: Water on its first condensation tends to unite in the form of hollow globules, or vesicles containing air; as it parts at the same time with its latent heat, the air, as well within the vesicles as between them, is rarefied, and the united mass of water and rarefied air may remain as light as an equal bulk of atmospheric air, or even lighter. Clouds may therefore remain floating in the atmosphere, or even rise. As this heat is dissipated, the clouds grow heavier and fall, while the air in the vesicles losing its elasticity, permits them to be broken by the internal pressure. The water then runs into drops, which, being many times heavier than atmospheric air, descend, forming Rain.

Clouds may be formed in all cases where the temperature of the ground is lower than that at which the vapour, mixed with atmospheric air, can remain permanent. Thus: whenever a warm wind flows over a cold surface, mists and fogs take place; and if the difference of temperature be considerable, they may break into rain. For an equal difference of temperature between the ground and air, it may be shown by calculations formed on the table of § 308, that the greatest quantity of precipitation will take place, when the two unequal temperatures are both high. Thus the causes that would produce heavy rains in warm climates, may produce no more than fogs, or dense mists, in those that are colder.

Clouds may also be formed on sudden changes of wind, upon the principle explained in § 491, when two masses of air are mixed that are both nearly saturated with moisture. It is to this cause that nearly all the rains of temperate climates are due. The passing of warm winds over cold surfaces, rarely produces more than mists or fogs, except in warm climates.

When clouds, after being formed, begin to descend, in consequence of the dissipation of the heat, by the rarefaction arising from which they are supported, they often reach strata of the atmosphere comparatively dry, and of higher temperature than they themselves possess. In such a case, the vapour may be again taken up, and the cloud dissipated. Thus clouds are frequently seen to roll down the sides of the mountains, and to disappear at a certain level; this is a proof of a dry state of the air beneath,

and is therefore considered by the inhabitants of mountainous countries, as a prognostic of fair weather.

When a cloud, on the other hand, formed in high and cold regions, passes in its descent through strata saturated with moisture, or nearly so, it may cool them until precipitation ensue; the precipitated moisture, uniting itself to the descending cloud, will augment the intensity of the rain it causes. Thus the same rain will be more copious in vallies, than upon the neighbouring mountains; and the difference is so sensible in this respect, that it has been detected by means of the rain-gauges at the observatory of Paris, one of which is upon the ground, the other upon the terraced roof of the building.

497. When the precipitation of vapour ensues at temperatures below the freezing point, Snow is formed; the particles of the condensed aqueous matter being free to move in any direction, arrange themselves under the action of their mutual attraction, in the manner of crystals. These crystals have usually the figure of six-pointed stars; and the aggregation of broken crystals of this shape forms flakes of snow.

498. Hail is a phenomenon that is not completely explained; the best theory on the subject, although not absolutely satisfactory, is as follows:

It is known that when water is frozen in a torricellian vacuum, it granulates and assumes the form of hail; hail also reaches the ground with a very great velocity: hence we may conclude, that it is formed in very rare air, and in a high region of the atmosphere. The decomposition of organic substances, is constantly giving out hydrogen gas, and this, from its specific levity, rises to the higher regions of the atmosphere; hence, as no gas can remain long over another unmixed, it mingles with atmospheric air, and becomes susceptible of being inflamed by electricity. Should it be thus acted upon, it forms water, will be condensed into a space much less than it formerly occupied, and would leave a vacuum, did not the adjacent portions of air rush in to fill the void. The sudden rarefaction of this air will produce an intense cold; the newly formed water will be frozen, and under circumstances that will cause it to granulate; descending from a lofty region, it will have great velocity; formed from hydrogen gas, and by the electric discharge, it will occur most frequently during the summer months, and accompany lightning.

499. It was shown in the preceding chapter, that the earth retains a constant mean temperature, under the joint action of solar and terrestrial radiation; but that the rate of these is unequal, not only at different seasons, but from hour to hour; the former ceases alto-

gether at the setting of the sun, while the latter continues for a time undiminished. Hence the surface of the earth cools rapidly after sunset, and may speedily reach the dew-point of the air in contact with it. So soon as this is the case, moisture begins to be precipitated, and a cloud is formed, the descent of the water of which this is composed, forms the deposit, that we call Dew. The cooling will be propagated slowly upwards, and the cloud will appear to rise; notwithstanding which, the moisture of which it is composed, actually falls. After some hours, the earth and air will assume the same temperature, and the cloud will disappear.

The first morning rays of the sun, passing horizontally through the air, will heat it, long before their influence can be felt upon the ground. The air will therefore acquire a greater capacity for moisture: if there be any water in the vicinity, vapour will rise, and propagate itself through the mass; but as the ground still remains colder, a new precipitation will ensue; thus dew will again be formed, and moisture occur in the morning.

500. When the surface of the ground, or of any other substance, is cooled by radiation to the temperature of 32°, the dew is frozen, and takes the form of white or hoar frost; this may often be deposited, when the temperature both of the air and of the ground at a very small depth, is above that of freezing.

501. When clouds exist in the atmosphere, the radiation is impeded, and dew will not be formed. Thus a want of dew is usually a prognostic of rain.

When the air is still, dew is most copious, and thus it falls in greatest abundance in sheltered situations, and frosts will continue later in the spring, and begin earlier in autumn, in vallies than on the open hills in the vicinity.

The motion of air mixes the portion cooled by contact with the earth, with that which is not, and brings new masses into contact; hence, although the loss of heat by radiation, may be as great or even greater, the ground will receive heat from the air, and the change of temperature will be less. In conformity, heavy dews do not fall during the prevalence of high winds, and hoar frosts rarely occur while they blow.

Surfaces that radiate well, will be most cooled, and will in consequence receive the greatest quantity of dew; and thus of land frequently tilled, and that which is left undisturbed, the latter will derive most moisture from the atmosphere in this form.

502. In the prosecution of this subject, we have in some degree trespassed upon the limits of pure physical science: this was however necessary, in order to give a correct view of the phenomena, but the discussion is incomplete from the propriety of confining ourselves as closely as possible to what is strictly mechanical.

So also in a variety of other cases, it has been judged expedient to extend our investigations into collateral branches of knowledge, while subjects of no small extent, and more immediately connected with Mechanics, have been passed over. This extension on the one hand, and omission on the other, have been determined by a view to the practical applications of our subject. These applications were originally intended to have formed a part of the work, and by means of them its true scope and objects would have become apparent. In its present form, however, the author trusts it will be found an useful introduction to the study of a most important and interesting branch of science, whether it be considered in its immediate connexion with the arts, or in its bearing upon knowledge of a more elevated character.

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